OPTIONAL STOCHASTIC INTEGRATION IN HILBERT SPACE WITH APPLICATIONS TO NUCLEAR SPACES

BY

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This work develops the theory of the stochastic integral of an optional process H with respect to a special semimartingale X. The theory is developed in three settings, with three distinct approaches to the construction of the integrals. In each setting, the integrals are first defined when X is a square integrable martingale and are then naturally extended to special semimartingales.

The first setting takes H to be real valued and X to be Hilbert valued. The approach in this case is similar to the classical compensated stochastic integral, when both processes are real valued, and uses the properties of the square bracket of Hilbert valued semimartingales.

In the second setting, H takes values in a separable Hilbert space and X is real valued. The integral is constructed, when X is a square integrable martingale, by means of an orthonormal expansion of H.

The third setting takes both H and X to be Hilbert valued. The approach in this case offers a more natural definition, when X is square integrable, by representing the optional stochastic integral in terms of a predictable stochastic integral. Moreover, this method may be applied to the previous two settings, in the case of bounded H, and in particular to the classical compensated stochastic integral.

All three approaches offer true compensated stochastic integrals which are extensions of the predictable stochastic integral and which agree with the classical integral when the Hilbert space is the real line. The properties of the integrals, including convergence theorems, are studied in detail in each setting.

The work concludes with the application of the three integrals to the theory of stochastic integration in nuclear spaces. Once again, the integrals are defined in three settings, which are analogous to the previously defined settings. The properties of these integrals are easily determined, by reducing to the properties of the optional integrals in Hilbert space.

CHAPTER I
INTRODUCTION

The modern theory of the stochastic integral has long been a much studied and useful concept in the field of probability. For example, the study of stochastic differential equations and Brownian motion fundamentally relies on the stochastic integral and in fact led to its development as a theory. There are numerous texts, papers, and lecture notes on stochastic integrals, dating as early as the work of Ito in 1944, and Meyer gives perhaps the most masterly exposition of the theory of real stochastic integration [4, 11]. Kunita [8] was the first to consider integrals of Hilbert-valued processes. The theory in Hilbert space has since been widely developed by Metivier and others [9, 10].

More recently, the theory has been developed for defining the stochastic integral of a wide class of Banach-valued predictable processes with respect to another wide class of Banach-valued processes. There are many different approaches to developing the stochastic integrals in these cases. For example, the theory of vector measures is a powerful tool in defining integrals with respect to operator-valued processes. This approach involves defining a measure m on an algebra of sets, which generates the predictable σ-algebra, and then obtaining a countably additive extension of m to the entire predictable σ-algebra. The role of the predictable σ-algebra is fundamental to this approach and in fact to the general development of the stochastic integral. In certain cases, the stochastic integral has been extended to include optional processes. The essential goal in this study is to make a complete extension of the stochastic integral in Hilbert space to optional processes.

In the case of real-valued processes, Déllacherie and Meyer [4] give the classical extension of the stochastic integral to optional processes. When integrating with respect to a square integrable martingale, they define the integral in terms of an operator on L^2. Using the theory of H^1 spaces and BMO spaces, they also make the extension for integrating with respect to a local martingale. We shall see that these methods extend naturally to defining the stochastic integral of a real optional process with respect to a Hilbert-valued special semimartingale.

Other less successful extensions to optional processes have also been made. By working with a continuous semimartingale X, Laurent Schwartz [13] was able to define a natural extension to optional processes. He approximates, in some sense, an optional process H by a predictable process H^1 and defines the integral of H with respect to X to be the integral of H^1 with respect to X. Similarly, Metivier [9] gives an example of extending his integral to include optional processes by integrating with respect to a semimartingale with a continuous control process. These extensions are natural and immediate; however, they rely on the very special continuity conditions. The essential problem is to define the integral of an optional process with respect to a noncontinuous semimartingale. We shall see that we may do so for special semimartingales.

The problem of defining the stochastic integral of a predictable process H with respect to a semimartingale X essentially reduces, in the real valued case, to the case when X is a square integrable martingale. One would like the integral, $H \cdot X$, of H with respect to X to be defined as naturally as possible. In particular, if X is a process of finite variation and H is bounded, we would like $H \cdot X$ to be the usual Stieltjes integral $\int^t H_s dX_s$. Moreover, if H is of the form $H = K_1 1_{]t_1,t_2]}$, where K_1 is bounded and F_{t_1}-measurable, we would like to define $H \cdot X$ by $H \cdot X = K_1(X^{t_2} - X^{t_1})$. We would also like the integral to have certain aesthetic properties, such as linearity in H and X, and to satisfy convergence theorems.

The method of vector measures is a natural, though not the classical, method of defining $H \cdot X$ for predictable H and square integrable X. We consider processes of the form

$$H = 1_{A_0} 1_{[0]} + 1_{B_0} 1_{]t_0,t_1]} + \cdots + 1_{B_n} 1_{]t_n,t_{n+1}]},$$

where A_0 is in F_0, B_i is in F_{t_i} for $i = 1,\ldots,n$, and $0 \leq t_0 < t_1 < \ldots < t_{n+1} < \infty$.

These processes form an algebra C and generate the predictable σ-algebra. We define a measure I_X on C by

$$I_X(H) = 1_{A_0} X_0 + 1_{B_0}(X_{t_0} - X_{t_1}) + \cdots + 1_{B_n}(X_{t_{n+1}} - X_{t_n}).$$

I_X is an L^2-valued, finitely additive measure on C. It can be shown that I_X has a unique countably additive extension to the predictable σ-algebra. Then for every bounded, predictable process H, we let $(H\cdot X)_t = \int H1_{[0,t]}dI_X$. In general, H·X is defined for any predictable process H that is integrable with respect to the measure I_X. This method extends naturally, though nontrivially, to Banach-valued and operator-valued processes. Schwartz [13, p. 66] has shown, however, that this method cannot be carried out for optional processes in an analogous manner, even in the real valued case. Thus, we also have no hope of using this method for optional processes in Hilbert space.

We now consider the problem of defining, in the most natural way, the stochastic integral $H \underset{c}{\cdot} X$ of an optional process H with respect to a semimartingale X, so that $H \underset{c}{\cdot} X = H\cdot X$ when H is predictable. We shall see in Chapter III that this integral cannot possibly agree with the Stieltjes integral when X is a process of finite variation, unless of course H is predictable. Yor [16] has defined one such integral H*X again in terms of a predictable stochastic integral. He approximates an optional process H, when possible, by a locally bounded predictable process H^1 such that $H - H^1$ is thin (or scanty) and defines H*X by

$$H*X = H^1\cdot X + \sum_{s\le t} (H_s - H^1_s)\Delta X_s,$$

provided the sum is absolutely convergent for every t. One sees immediately that this integral agrees with that of Schwartz, mentioned above, when X is continuous. However, unless X is of finite variation, one cannot verify that even bounded H are integrable in this sense. The second goal in this study is to give a natural definition of the stochastic integral of a wide class of optional processes, in terms of the predictable stochastic integral, in both the real-valued case and in Hilbert space. The outgrowth of this research stemmed from joining Professor James K. Brooks in his study of stochastic integration of abstract processes.

We shall, therefore, develop the theory of the stochastic integral $H \underset{c}{\cdot} X$ of an optional process H with respect to a special semimartingale X in three distinct settings. In each case, the class of processes H for which we define the integral will include all locally bounded processes. Moreover, $H \underset{c}{\cdot} X$ will agree with the predictable stochastic integral $H \cdot X$ when H is predictable. In each setting, the integrals are first defined when X is a square integrable martingale and are then naturally extended to special semimartingales. The properties of the integrals are studied in detail in each setting. This work is as self-contained as practically possible and contains a chapter on the predictable stochastic integral in Hilbert space. This chapter also contains the fundamental theorems on the square bracket $[X, Y]$ of Hilbert-valued semimartingales X and Y and H^p and BMO^p spaces of Hilbert-valued martingales.

In our first setting for defining the optional integral, Chapter III, we take H to be real-valued while X takes values in a separable

Hilbert space. $H \underset{c}{\cdot} X$ will be a Hilbert-valued semimartingale and will be a square integrable martingale if X is. The approach in this setting is similar to the classical optional (also called compensated) stochastic integral for real processes given by Déllacherie and Meyer [4].

In Chapter IV, we shall let X be real-valued while H takes values in a separable Hilbert space. Again $H \underset{c}{\cdot} X$ will be a Hilbert-valued semimartingale and if X is a square integrable martingale, then so is $H \underset{c}{\cdot} X$. In this setting, we shall use an alternate approach for constructing the integral. When X is square integrable though, we may use an orthonormal expansion of H to reduce to the real-valued case.

Finally, in Chapter V, we consider the case when both H and X are Hilbert-valued. We begin by letting H be bounded and X be a square integrable martingale. The approach in this case gives us the natural definition of $H \underset{c}{\cdot} X$ in terms of the predictable stochastic integral. This method applies, in particular, to the real-valued case and we shall show that it may also be used to define $H \underset{c}{\cdot} X$, when H is bounded, in the previous two settings.

We conclude by applying the three integrals to the theory of stochastic integration in nuclear spaces. Once again, we shall define integrals in three settings, which are analogous to the previously discussed settings. The properties of these integrals will be easily determined, by reducing to the properties of the optional integrals in Hilbert space.

Throughout this manuscript, we work with an underlying probability space (Ω, F, P) with a filtration (F_t) that satisfies the usual conditions. Moreover, we assume that $F_{0-} = F_0$ and $F_{\infty-} = F_\infty$. The space E will always denote a real, separable Hilbert space. In Chapters II-V, the processes with which we work will always be either real-valued or E-valued. We point out that all semimartingales (in particular, local martingales, martingales, and processes of finite variation) are adapted and cadlag. Finally, we shall denote by M_E^2 (resp. M^2) the space of E-valued (resp. real-valued) square integrable martingales which, by identifying such a martingale with its limit at infinity, is isometrically isomorphic to $L^2(\Omega, F, P, E)$ (resp. $L^2(\Omega, F, P, R)$).

CHAPTER II
THE PREDICTABLE STOCHASTIC INTEGRAL, SQUARE BRACKET, AND APPLICATIONS

This chapter contains fundamental theorems on the predictable stochastic integral for real and Hilbert-valued processes and the square bracket $[X, Y]$ for Hilbert-valued semimartingales X and Y. These classical results appear in much the same form in many books and papers [see 3, 4, 8, 9, 10, and 11] and proofs are given only for selected theorems in which the classical proofs for real-valued processes do not extend naturally to Hilbert space. We also give applications to H^p and BMO spaces of Hilbert-valued processes, again to which many of the classical proofs from Déllacherie and Meyer [4] still apply.

1. The Predictable Stochastic Integral

In this section, we give a list of theorems on the predictable stochastic integral $H{\cdot}X$ of a predictable process H with respect to a square integrable martingale X. H and X may be either real or Hilbert valued. Juxtaposition of processes will denote either the product, scalar product, or inner product depending on whether the corresponding processes are real and/or Hilbert-valued. The symbol $|\cdot|$ will denote either absolute value or the norm of the separable Hilbert space E and $(\ ,\ )$ will denote the inner product in E.

1 DEFINITION. Let X be a square integrable martingale. $L_p^2(X)$ is the space of predictable processes H such that

$$\|H\|_X = (E[\int^\infty |H_s|^2 d\langle X, X\rangle_s])^{1/2} < \infty.$$

(Recall that $\langle X, X\rangle$ is the unique real increasing predictable process such that $|X_t|^2 - \langle X, X\rangle_t$ is a uniformly integrable martingale which is zero at 0.)

2 DEFINITION. An elementary predictable process H is a process of the form

$$H = K_{-1}1_{[0]} + K_0 1_{]t_0,t_1]} + \cdots + K_n 1_{]t_n,t_{n+1}]}$$

where K_i are bounded, F_{t_i}-measurable, $(F_{t_{-1}} = F_0)$ and $0 \leq t_0 < t_1 < \ldots < t_{n+1} < \infty$.

3 THEOREM. Elementary predictable processes are dense in $L_p^2(X)$.

Sketch of Proof: We first remark that we may replace $\langle X, X\rangle$ in the definition of $L_p^2(X)$ by any integrable increasing process and the theorem remains valid. By truncating, it suffices to prove the result for bounded H. If H is real-valued, the result is true by a monotone class argument, since the elementary predictable processes generate the predictable σ-field.

For H E-valued, let $\{e_n\}$ be an orthonormal basis of E. Then $H = \sum_n H^n e_n$, where $H^n = (H, e_n)$, and $|H_s|^2 = \sum_n |H_s^n|^2$. Let $\varepsilon > 0$ be given.

For every n, there exists an elementary process L^n such that

$$E[\int^\infty |H^n_s - L^n_s|^2 d\langle X, X\rangle_s] < \frac{\varepsilon}{2^n}.$$

Let $L = \sum_n L^n e_n$. L is now E-valued and elementary and

$$E[\int^\infty |H_s - L_s|^2 d\langle X, X\rangle_s] = E[\int^\infty \sum_n |H^n_s - L^n_s|^2 d\langle X, X\rangle_s]$$

$$= \sum_n E[\int^\infty |H^n_s - L^n_s|^2 d\langle X, X\rangle_s] < \varepsilon.$$

4 THEOREM. Let one of H and X be real-valued. If H is in $L^2_p(X)$, then there exists a unique square integrable martingale $H \cdot X$, called the stochastic integral of H with respect to X, such that for every square integrable martingale Y, having the same range as X,

$$E[(H \cdot X)_\infty Y_\infty] = E[\int^\infty H_s d\langle X, Y\rangle_s].$$

If H is elementary of the form in Definition 2, then

$$(4.1) \quad H \cdot X = K_{-1} X_0 + K_0(X^{t_1} - X^{t_0}) + \ldots + K_n(X^{t_{n+1}} - X^{t_n}).$$

5 THEOREM. Let H and X be E-valued. If H is in $L^2_p(X)$, then there exists a unique real square integrable martingale $H \cdot X$, called the stochastic integral of H with respect to X, such that for every real square integrable martingale Y

$$E[(H \cdot X)_\infty Y_\infty] = E[\int^\infty H_s dA^Y_s]$$

where A^Y is the E^*(=E)-valued process of integrable variation defined by

$$(\phi, A_t^Y(w)) = < Y, (\phi, X)>_t(w), \phi \text{ in } E.$$

Moreover, the processes $<H \cdot X, Y>$ and $\int^t H_s dA_s^Y$ are indistinguishable.

Proof: The uniqueness is obvious for if another such square integrable martingale K existed, we would have that, for every Y_∞ in $L^2(\Omega, F, P, R)$,

$$E[(H \cdot X)_\infty Y_\infty] = E[K_\infty Y_\infty]$$

and hence that $(H \cdot X)_\infty = K_\infty$ a.s. It follows that $H \cdot X$ and K are indistinguishable.

Kunita [8] gives the classical construction of $H \cdot X$. If H is elementary, in the form of Definition 2, then

$$H \cdot X = (K_{-1}, X_0) + \dots + (K_n, X^{t_{n+1}} - X^{t_n}).$$

$H \cdot X$ is a square integrable martingale and

$$\|(H \cdot X)_\infty\|_{L^2} \leq \|H\|_X.$$

If H is in $L_p^2(X)$, then $H \cdot X$ is the limit in M^2 of the sequence $(H^n \cdot X)$, where (H^n) is any sequence of elementary predictable processes converging to H in $L_p^2(X)$. The mapping $H \to H \cdot X$, from $L_p^2(X)$ into M^2, is a linear contraction.

Before continuing with the proof, we list some basic properties of the sharp bracket.

If U and V are E-valued square integrable martingales and S and T are stopping times with $S \leq T$, then

(5.1) $$\langle U^T - U^S, V^T - V^S\rangle = \langle U, V\rangle^T - \langle U, V\rangle^S$$ and

(5.2) $$\int^\infty |d\langle U, V\rangle_s| \leq \left(\int^\infty d\langle U, U\rangle_s\right)^{1/2}\left(\int^\infty d\langle V, V\rangle_s\right)^{1/2} \text{ a.s.}$$

which follows from the Kunita-Watanabe inequality, and if U is real-valued, then for every ϕ in E

(5.3) $$\langle U\phi, U\phi\rangle = |\phi|^2\langle U, U\rangle$$ and

(5.4) $$\langle U, (\phi, V)\rangle = \langle U\phi, V\rangle.$$

Now let Y be a real square integrable martingale. We shall show A^Y is of integrable variation. We suppress the Y in the notation. Let $\pi_n = \{t_0, t_1, \ldots, t_{k_n}\}$ be an increasing sequence of partitions of $[0, \infty]$ such that $\lim_{n\to\infty} \text{mesh}(\pi_n) = 0$. Then

$$\int^\infty |dA_s| = \sup \sum_{i=0}^{k_n - 1} |A_{t_{i+1}} - A_{t_i}|$$

$$= \sup_n \sum_{i=0}^{k_n - 1} \Big(\sup_{\substack{\phi\varepsilon E \\ |\phi|\leq 1}} |(\phi, A_{t_{i+1}}) - (\phi, A_{t_i})|\Big)$$

$$= \sup_n \sum_{i=0}^{k_n - 1} \Big(\sup_{\substack{\phi\varepsilon E \\ |\phi|\leq 1}} |\langle Y\phi, X\rangle_{t_{i+1}} - \langle Y\phi, X\rangle_{t_i}|\Big)$$

$$= \sup_n \sum_{i=0}^{k_n - 1} \Big(\sup_{\substack{\phi\varepsilon E \\ |\phi|\leq 1}} |\langle Y^{t_{i+1}}\phi - Y^{t_i}\phi, X^{t_{i+1}} - X^{t_i}\rangle_\infty|\Big)$$

$$\leq \sup_n \sum_{i=0}^{k_n-1} \Big(\sup_{\substack{\phi \varepsilon E \\ |\phi| \leq 1}} \int^\infty |d\langle Y^{t_{i+1}}\phi - Y^{t_i}\phi, X^{t_{i+1}} - X^{t_i}\rangle_s| \Big)$$

$$\leq \sup_n \sum_{i=0}^{k_n-1} \langle Y^{t_{i+1}} - Y^{t_i}, Y^{t_{i+1}} - Y^{t_i}\rangle_\infty^{\frac{1}{2}} \langle X^{t_{i+1}} - X^{t_i}, X^{t_{i+1}} - X^{t_i}\rangle_\infty^{\frac{1}{2}}$$

$$= \sup_n \sum_{i=0}^{k_n-1} (\langle Y, Y\rangle_{t_{i+1}} - \langle Y, Y\rangle_{t_i})^{\frac{1}{2}} (\langle X, X\rangle_{t_{i+1}} - \langle X, X\rangle_{t_i})^{\frac{1}{2}}$$

$$\leq \sup_n \Big(\sum_{i=0}^{k_n-1} \langle Y, Y\rangle_{t_{i+1}} - \langle Y, Y\rangle_{t_i} \Big)^{\frac{1}{2}} \Big(\sum_{i=0}^{k_n-1} \langle X, X\rangle_{t_{i+1}} - \langle X, X\rangle_{t_i} \Big)^{\frac{1}{2}}$$

$$\leq \langle Y, Y\rangle_\infty^{\frac{1}{2}} \langle X, X\rangle_\infty^{\frac{1}{2}} .$$

Thus, A^Y is of integrable variation, since $\langle Y, Y\rangle_\infty^{1/2} \langle X, X\rangle_\infty^{1/2}$ is integrable.

We next note that if Z is a real square integrable martingale that is zero on $[0, t_0]$, then

$$(YZ - \langle Y, Z\rangle)^{t_0} = YZ^{t_0} - \langle Y, Z^{t_0}\rangle = 0.$$

That is, $YZ - \langle Y, Z\rangle$ is a uniformly integrable martingale which is also zero on $[0, t_0]$. Thus, for any F_{t_0}-measurable, bounded, random variable H, $YHZ - H\langle Y, Z\rangle$ is a uniformly integrable martingale which is zero at 0. Thus,

(5.5) $$\langle Y, HZ\rangle = H\langle Y, Z\rangle.$$

Next, suppose that Z is an E-valued square integrable martingale which is zero on $[0, t_0]$. Let B^Z be the E-valued process of integrable variation given by

$$(\phi, B_t^Z(w)) = \langle Y, (\phi, Z)\rangle_t(w), \ \phi \text{ in } E.$$

Then for every F in F_{t_0} and for every ϕ in E, (ϕ, Z) is real valued; thus, by (5.5),

$$(1_F\phi, B_t^Z) = 1_F(\phi, B_t^Z) = 1_F\langle Y, (\phi, Z)\rangle_t$$

$$= \langle Y, 1_F(\phi, Z)\rangle_t = \langle Y, (1_F\phi, Z)\rangle_t.$$

It follows that if K is a simple, F_{t_0}-measurable, E-valued, random vector of the form $K = \sum_{i=1}^{n} 1_{F_i}\phi_i$, with F_i in F_{t_0} and ϕ_i in E, then

(5.6) $$(K, B^Z) = \langle Y, (K, Z)\rangle.$$

If K is a bounded, F_{t_0}-measurable, random vector, then there exists a sequence (K^n) of simple, F_{t_0}-measurable, random vectors such that $\lim_{n\to\infty} K^n = K$ a.s. and $|K^n| \le |K|$ a.s. for all n. By the dominated convergence theorem, it follows that (K^n, Z_∞) converges to (K, Z_∞) in L^2; hence, (K^n, Z) converges to (K, Z) in M^2. We then have for every t

$$E[|\langle Y, (K^n, Z)\rangle_t - \langle Y, (K, Z)\rangle_t|]$$

$$= E[|\int^t d\langle Y, (K^n - K, Z)\rangle_s|]$$

$$\le E[\int^\infty |d\langle Y, (K^n - K, Z)\rangle_s|]$$

$$\le E[(\int^\infty d\langle Y, Y\rangle_s)^{1/2}(\int^\infty d\langle (K^n - K, Z), (K^n - K, Z)\rangle_s)^{1/2}]$$

$$\leq (E[\langle Y, Y\rangle_\infty])^{1/2}(E[\langle (K^n - K, Z), (K^n - K, Z)\rangle_\infty])^{1/2}$$

$$= \|Y_\infty\|_{L^2}\|(K^n - K, Z_\infty)\|_{L^2}.$$

The second inequality is by (5.2) and the third is Holder's inequality. Thus, $\langle Y, (K^n, Z)\rangle_t$ converges to $\langle Y, (K, Z)\rangle_t$ in L^1. But, by dominated convergence, $\langle Y, (K^n, Z)\rangle_t = (K^n, B^Z_t)$ converges to (K, B^Z_t) in L^1. Hence, for every t, $\langle Y, (K, Z)\rangle_t = (K, B^Z_t)$ a.s. By right continuity, we have (up to an evanescent set)

(5.7) $$\langle Y, (K, Z)\rangle = (K, B^Z).$$

We are now prepared to prove the characterization of $H\cdot X$. First, suppose H is elementary of the form $H = K_{-1}1_{[0]} + K_0 1_{]t_0,t_1]} + \dots + K_n 1_{]t_n,t_{n+1}]}$. Then $H\cdot A = (K_{-1}, A_0) + (K_0, A^{t_1} - A^{t_0}) + \dots + (K_n, A^{t_{n+1}} - A^{t_n})$ and $H\cdot X = (K_{-1}, X_0) + (K_0, X^{t_1} - X^{t_0}) + \dots + (K_n, X^{t_{n+1}} - X^{t_n})$. $Z^i = X^{t_{i+1}} - X^{t_i}$ is zero on $[0, t_i]$, so by (5.7), we have

(5.8) $$(K_i, B^{Z_i}) = \langle Y, (K, Z^i)\rangle, \ i = 0,\dots,n.$$

However, $A^{t_{i+1}} - A^{t_i} = \langle Y, (\cdot, X)\rangle^{t_{i+1}} - \langle Y, (\cdot, X)\rangle^{t_i}$

$$= \langle Y, (\cdot, X^{t_{i+1}} - X^{t_i})\rangle$$

$$= \langle Y, (\cdot, Z^i)\rangle$$

$$= B^{Z^i}.$$

Similarly, for $Z = X^0 = X_0$, we have that $A^0 = B^Z$ and hence that $(K_{-1}, A_0) = \langle Y, (K_{-1}, X_0)\rangle$. Together with (5.8) this implies

$$H\cdot A^Y = \langle Y, H\cdot X\rangle. \tag{5.9}$$

In particular, $E[\int^\infty H_s dA_s^Y] = E[\langle Y, H\cdot X\rangle_\infty]$

$$= E[Y_\infty (H\cdot X)_\infty].$$

Now let H be bounded and predictable. Then $E[\int^\infty |H_s|^2 (d\langle X, X\rangle_s + |dA_s|)] < \infty$. By Theorem 3, there exists a sequence (H^n) of elementary predictable processes such that

$$\lim_{n\to\infty} E[\int^\infty |H_s^n - H_s|^2 (d\langle X, X\rangle_s + |dA_s^Y|)] = 0.$$

Then (H^n) converges to H in $L_p^2(X)$ and hence $(H^n\cdot X)$ converges to $H\cdot X$ in M^2, since the mapping $H \to H\cdot X$ is a contraction. By (5.2) and Holder's inequality, we have for every t that $\langle Y, H^n\cdot X\rangle_t$ converges to $\langle Y, H\cdot X\rangle_t$ in L^1. Since

$$|\int^t H_s^n dA_s^Y - \int^t H_s dA_s^Y|^2 \le \int^t |H_s^n - H_s|^2 |dA_s^Y|,$$

we have that $\langle Y, H^n\cdot X\rangle_t = \int^t H_s^n dA_s^Y$ converges in L^2, and thus in L^1, to $\int^t H_s dA_s^Y$.

It follows that for every t,

$$\int^t H_s dA_s^Y = \langle Y, H\cdot X\rangle_t \quad \text{a.s.}$$

By right continuity, the processes are indistinguishable.

Finally, let H be in $L^2_p(X)$. Let $H^n = H1_{\{|H| \le n\}}$. Each H^n is bounded and by dominated convergence, (H^n) converges to H in $L^2_p(X)$. Hence, $(H^n \cdot X)$ converges to $H \cdot X$ in M^2. Again by (5.2) and Holder's inequality, we have for every t that $\langle Y, H^n \cdot X\rangle_t$ converges to $\langle Y, H \cdot X\rangle_t$ in L^1.

The processes $\int^t H^n_s dA^Y_s$ and $\langle Y, H^n \cdot X\rangle_t$ are equal and thus have the same variation, $\int^t |H^n_s||dA^Y_s|$ and $\int^t |d\langle Y, H^n \cdot X\rangle_s|$ respectively. It follows again by (5.2) that

$$E[|\int^\infty |H^n_s||dA^Y_s| - \int^\infty |H^m_s||dA^Y_s||] \le E[\int^\infty |H^n_s - H^m_s||dA^Y_s|]$$

$$= E[\int^\infty |d\langle Y, (H^n - H^m)\cdot X\rangle_s|]$$

$$\le \|Y_\infty\|_{L^2}\|(H^n - H^m)\cdot X\|_{M^2}.$$

Thus, the sequence $(\int^\infty |H^n_s||dA^Y_s|)$ is Cauchy in $L^1(\Omega, F, P)$, which is complete. Its pointwise limit must then also be in L^1. Since $(|H^n|)$ increases to $|H|$, we have by monotone convergence that $\int^\infty |H^n_s||dA^Y_s|$ converges to $\int^\infty |H_s||dA^Y_s|$ for a.e. w and hence in $L^1(\Omega, F, P)$. Moreover, since (H^n) converges to H pointwise, we have

$$\lim_{n\to\infty} \int^\infty |H^n_s - H_s||dA^Y_s| = 0 \text{ for a.e. w.} \tag{5.10}$$

Since $\int^\infty |H_s||dA^Y_s|$ is integrable, we have for a.e. w and for all t that

$$\int^t |H_s||dA^Y_s| \le \int^\infty |H_s||dA^Y_s| < \infty.$$

So the process $\int^t H_s dA^Y_s$ is defined (outside of an evanescent set). Then for every t,

$$\left|\int^t H^n_s dA^Y_s - \int^t H_s dA^Y_s\right| \leq \int^\infty |H^n_s - H_s||dA^Y_s|.$$

So for a.e. w, $\int^t H_s dA^Y_s = \lim_{n\to\infty} \int^t H^n_s dA^Y_s$ for all t.

Since $\int^t H^n_s dA^Y_s = \langle Y, H^n\cdot X\rangle_t$ converges in L^1 to $\langle Y, H\cdot X\rangle_t$, we have that for every t

$$\int^t H_s dA^Y_s = \langle Y, H\cdot X\rangle_t \qquad \text{a.s.}$$

By right continuity, the processes are indistinguishable. In particular, $E[\int^\infty H_s dA^Y_s] = E[\langle Y, H\cdot X\rangle_\infty]$

$$= E[Y_\infty (H\cdot X)_\infty].$$

6 THEOREM. Let (H^n) converge to H in $L^2_p(X)$. There exists a subsequence (n_i) such that for almost every w, $(H^{n_i}\cdot X)_s$ converges to $(H\cdot X)_s$ uniformly in s.

7 THEOREM. Let H be in $L^2_p(X)$. If H is E-valued and X is real, or if H is real and X is E-valued, then $\langle H\cdot X, H\cdot X\rangle = |H|^2\cdot\langle X, X\rangle$.

Proof: The proof is the same in either case. It suffices to show $|H\cdot X|^2 - |H|^2\cdot\langle X, X\rangle$ is a uniformly integrable martingale which is zero at 0. The result will then follow from the uniqueness of the sharp bracket. First let H be elementary of the form $H = K_{-1}1_{[0]} + K_0 1_{]t_0,t_1]} + \cdots + K_n 1_{]t_n,t_{n+1}]}$. Then

$$(7.1)\qquad |H\cdot X|^2 = |K_{-1}X_0|^2 + \sum_{i=0}^{n} |K_i(X^{t_{i+1}} - X^{t_i})|^2$$

$$+ 2\sum_{i=0}^{n} (K_i(X^{t_{i+1}} - X^{t_i}), K_{-1}X_0)$$

$$+ \sum_{i\neq j} (K_i(X^{t_{i+1}} - X^{t_i}), K_j(X^{t_{j+1}} - X^{t_j})), \quad \text{and}$$

$$(7.2)\qquad |H|^2\cdot\langle X, X\rangle = |K_{-1}|^2\langle X, X\rangle_0$$

$$+ \sum_{i=0}^{n} |K_i|^2(\langle X, X\rangle^{t_{i+1}} - \langle X, X\rangle^{t_i}).$$

$X^{t_{i+1}} - X^{t_i}$ is a square integrable martingale which is zero on $[0, t_i]$; thus, so is $K_i(X^{t_{i+1}} - X^{t_i})$, since K_i is bounded and F_{t_i}-measurable. Since $K_{-1}X_0$ is a square integrable F_0-measurable random vector, $(K_i(X^{t_{i+1}} - X^{t_i}), K_{-1}X_0)$ remains a uniformly integrable martingale, for each $i=0,\ldots,n$.

For $i\neq j$, $(X^{t_{i+1}} - X^{t_i})(X^{t_{j+1}} - X^{t_j})$ is a uniformly integrable martingale, right closed by $(X_{t_{i+1}} - X_{t_i})(X_{t_{j+1}} - X_{t_j})$, and is zero on $[0, t_i \vee t_j]$. Since K_iK_j is bounded and $F_{t_i \vee t_j}$-measurable,

$$K_iK_j(X^{t_{i+1}} - X^{t_i})(X^{t_{j+1}} - X^{t_j}) = (K_i(X^{t_{i+1}} - X^{t_i}), K_j(X^{t_{j+1}} - X^{t_j}))$$

remains a uniformly integrable martingale, for each $i=0,\ldots,n$.

Since $|K_{-1}|^2\langle X, X\rangle_0 = |K_{-1}|^2|X_0|^2 = |K_{-1}X_0|^2$, by subtracting (7.2) from (7.1), it remains to be shown that

$$|K_i|^2[(X^{t_{i+1}} - X^{t_i})^2 - (\langle X, X\rangle^{t_{i+1}} - \langle X, X\rangle^{t_i})]$$

is a uniformly integrable martingale, which is zero at 0. This follows from the definition of the sharp bracket; since

$$\langle X^{t_{i+1}} - X^{t_i}, X^{t_{i+1}} - X^{t_i}\rangle = \langle X, X\rangle^{t_{i+1}} - \langle X, X\rangle^{t_i},$$

$$(X^{t_{i+1}} - X^{t_i})^2 - (\langle X, X\rangle^{t_{i+1}} - \langle X, X\rangle^{t_i})$$

is a uniformly integrable martingale which is zero on $[0, t_i]$. The result follows since $|K_i|^2$ is bounded and F_{t_i}-measurable.

For H in $L_p^2(X)$, let (H^n) be a sequence of elementary processes converging to H in $L_p^2(X)$. Then $Y^n = |H^n \cdot X|^2 - |H^n|^2 \cdot \langle X, X\rangle$ is a uniformly integrable martingale. Let $Y = |H \cdot X|^2 - |H|^2 \cdot \langle X, X\rangle$.

For every $t \leq \infty$ and every A in F_t,

$$\int_A \int^t |H_s^n - H_s|^2 d\langle X, X\rangle_s dP \leq \|H^n - H\|_X^2; \text{ hence,}$$

$$\lim_{n\to\infty} \int_A \int^t |H_s^n|^2 d\langle X, X\rangle_s dP = \int_A \int^t |H_s|^2 d\langle X, X\rangle_s dP.$$

Moreover, $$|((H^n - H)\cdot X)_t|_{L^2}^2 = E[\int^t |H_s^n - H_s|^2 d\langle X, X\rangle_s]$$

$$\leq \|H^n - H\|_X^2.$$

Hence, $$\lim_{n\to\infty} \int_A |(H^n \cdot X)_t|^2 dP = \int_A |(H\cdot X)_t|^2 dP.$$

Thus, $$\int_A Y_\infty dP = \lim_{n\to\infty} \int_A Y_\infty^n dP = \lim_{n\to\infty} \int_A Y_t^n dP = \int_A Y_t dP.$$

That is, $E(Y_\infty | f_t) = Y_t$ a.s. and Y is a uniformly integrable martingale. Since $(|H|^2 \cdot \langle X, X\rangle)_0 = |H_0|^2 |X_0|^2 = |(H\cdot X)_0|^2$, Y is zero at 0. Hence,

$$\langle H\cdot X, H\cdot X\rangle = |H|^2\cdot\langle X, X\rangle.$$

2. Properties and More Theorems

Keeping the notation of Section 1, we shall list some properties of $H\cdot X$ and then extend the predictable stochastic integral for a locally bounded, predictable process H with respect to a semimartingale X.

1 THEOREM.

(a) If H is in $L_p^2(X)$ and $L_p^2(Y)$, then H is in $L_p^2(X + Y)$ and $H\cdot(X + Y) = H\cdot X + H\cdot Y$.

(b) Let H and K be in $L_p^2(X)$. Then $H + K$ is in $L_p^2(X)$ and $(H + K)\cdot X = H\cdot X + K\cdot X$.

2 THEOREM. Let H be in $L_p^2(X)$.

(a) $\|H\cdot X\|_{M^2} = \|(H\cdot X)_\infty\|_{L^2} \leq \|H\|_X$, with equality if one of H or X is real-valued.

(b) $\Delta(H\cdot X) = H\Delta X$; $(H\cdot X)_0 = H_0X_0$.

(c) $(H\cdot X)^T = H\cdot X^T = H1_{[0,T]}\cdot X$, for every stopping time T.

(d) If X is also a process of finite variation, then $(H\cdot X)_t = \int^t H_s dX_s$, for locally bounded H.

3 REMARKS. Let X be a semimartingale and let H be a bounded, predictable process. We may write $X = X_0 + M + A$, where M is a local martingale with bounded jumps, A is a process of finite variation,

and $M_0 = 0 = A_0$. M is then locally square integrable; that is, there exists a sequence of stopping times (T_n) which increase to ∞ such that M^{T_n} is a square integrable martingale for each n. By Theorem 2(c), $(H\cdot M^{T_{n+1}})^{T_n} = H\cdot M^{T_n}$. We may then define $H\cdot M$ by pasting together. We let $H\cdot M = \lim_{n\to\infty} H\cdot M^{T_n}$, so that $(H\cdot M)^{T_n} = H\cdot M^{T_n}$. $H\cdot M$ is well-defined, independent of the choice of (T_n). Moreover, for every stopping time T

$$(H\cdot M)^T = H1_{[0,T]}\cdot M = H\cdot M^T. \tag{3.1}$$

We define $H\cdot A$ to be the Stieltjes integral $(H\cdot A)_t = \int^t H_s dA_s$. Again for every stopping time T

$$(H\cdot A)^T = H1_{[0,T]}\cdot A = H\cdot A^T. \tag{3.2}$$

Finally, we define $H\cdot X = H_0X_0 + H\cdot M + H\cdot A$. By localizing and applying Theorems 1(a) and 2(d), we see that $H\cdot X$ is well-defined, independent of the decomposition of X. Moreover, for every stopping time T

$$(H\cdot X)^T = H1_{[0,T]}\cdot X = H\cdot X^T. \tag{3.3}$$

4 REMARKS. If H is locally bounded and zero at 0, then there exists a sequence of stopping times (T_n), which increase to ∞, such that $H1_{[0,T_n]}$ is bounded for each n. By (3.3), we may define $H\cdot X$ by pasting together. We let $H\cdot X = \lim_{n\to\infty} H1_{[0,T_n]}\cdot X$, so that $(H\cdot X)^{T_n} = H1_{[0,T_n]}\cdot X$. $H\cdot X$ is well-defined, independent of the choice of (T_n). If H is not zero at 0, we define $H\cdot X = H_0X_0 + H1_{]0,\infty[}\cdot X$.

5 THEOREM. Let H be a locally bounded, predictable process and let X be a semimartingale.

(a) $H \cdot X$ is a semimartingale.

(b) If X is a local martingale, then so is $H \cdot X$.

(c) If X is a process of finite variation, then so is $H \cdot X$. Moreover, $(H \cdot X)_t = \int^t H_s dX_s$.

(d) If H is elementary, then $H \cdot X$ is given by formula 1.4.(4.1).

(e) If K is locally bounded and predictable, then $(H + K) \cdot X = H \cdot X + K \cdot X$.

(f) If Y is a semimartingale, then $H \cdot (X + Y) = H \cdot X + H \cdot Y$.

(g) $\Delta(H \cdot X) = H\Delta X$; $(H \cdot X)_0 = H_0 X_0$.

(h) For every stopping time T, $(H \cdot X)^T = H1_{[0,T]} \cdot X = H \cdot X^T$.

6 THEOREM.

(a) Let H be an E-valued, locally bounded, predictable process and let X be a real semimartingale. Then for every f in E, $(f, H \cdot X) = (f, H) \cdot X$.

(b) If H is real-valued and X is E-valued, then for every f in E, $(f, H \cdot X) = H \cdot (f, X)$.

Proof:

(a) Assume first that X is a square integrable martingale and H is in $L^2_p(X)$. Then for every real square integrable martingale Y, we have

$$E[(f, (H \cdot X)_\infty)Y_\infty] = E[(f, (H \cdot X)_\infty Y_\infty)]$$

$$= (f, E[(H \cdot X)_\infty Y_\infty])$$

$$= (f, E[\int^\infty H_s d\langle X, Y\rangle_s])$$

$$= E[\int^\infty (f, H_s) d\langle X, Y\rangle_s]$$

$$= E[((f, H)\cdot X)_\infty Y_\infty].$$

Hence, $(f, (H\cdot X)_\infty) = ((f, H)\cdot X)_\infty$ a.s., which implies $(f, H\cdot X)$ and $(f, H)\cdot X$ are indistinguishable.

If M is a locally square integrable local martingale which is zero at 0 and H is bounded, we choose stopping times T_n which increase to ∞ such that M^{T_n} is a square integrable martingale for each n. Then, outside of an evanescent set,

$$(f, H\cdot M) = \lim_{n\to\infty}(f, H\cdot M^{T_n}) = \lim_{n\to\infty}(f, H)\cdot M^{T_n} = (f, H)\cdot M.$$

If A is a process of finite variation, then

$$(f, H\cdot A) = (f, \int^t H_s dA_s) = \int^t (f, H_s) dA_s = (f, H)\cdot A.$$

Decomposing X as $X = X_0 + M + A$, we then have for bounded H

$$(f, H\cdot X) = (f, H_0 X_0 + H\cdot M + H\cdot A)$$

$$= (f, H_0)X_0 + (f, H\cdot M) + (f, H\cdot A)$$

$$= (f, H_0)X_0 + (f, H)\cdot M + (f,H)\cdot A$$

$$= (f, H)\cdot X.$$

If H is locally bounded and zero at 0, we let stopping times T_n increase to ∞ such that $H1_{[0,T_n]}$ is bounded for each n. Then, outside of an evanescent set,

$$(f,\ H\cdot X) = \lim_{n\to\infty}(f,\ H1_{[0,T_n]}\cdot X) = \lim_{n\to\infty}(f,\ H1_{[0,T_n]})\cdot X$$
$$= (f,\ H)\cdot X.$$

If H is not zero at 0, then

$$(f,\ H\cdot X) = (f,\ H_0X_0) + (f,\ H1_{]0,\infty[}\cdot X)$$
$$= (f,\ H_0)X_0 + (f,\ H1_{]0,\infty[})\cdot X$$
$$= (f,\ H)\cdot X.$$

(b) We first note that if X is an E-valued square integrable martingale and Y is a real square integrable martingale, then for every f in E, $(X,\ fY) = (f,\ X)Y$. Thus, $(X,\ fY) - \langle (f,\ X)\ Y\rangle = (f,\ X)Y - \langle (f,\ X),\ Y\rangle$ is a uniformly integrable martingale which is zero at 0. Hence,

$$\langle X,\ fY\rangle = \langle (f,\ X),\ Y\rangle. \tag{6.1}$$

Then, if H is in $L^2_p(X)$,

$$E[(f,\ (H\cdot X)_\infty)Y_\infty] = E[(fY_\infty,\ (H\cdot X)_\infty)]$$
$$= E[\int^\infty H_s d\langle X,\ fY\rangle_s]$$
$$= E[\int^\infty H_s d\langle (f,\ X),\ Y\rangle_s]$$
$$= E[(H\cdot (f,\ X))_\infty Y_\infty].$$

Hence, $(f, (H\cdot X)_\infty) = (H\cdot(f, X))_\infty$ a.s. and $(f, H\cdot X)$ is indistinguishable from $H\cdot(f, X)$.

As in part (a), the result remains true for a locally square integrable local martingale which is zero at 0, provided H is bounded.

If A is a process of finite variation, then

$$(f, H\cdot A) = (f, \int^t H_s dA_s) = \int^t H_s d(f, A_s) = H\cdot(f,A).$$

As in part (a), we obtain the result for a semimartingale and bounded H and then for locally bounded H.

7 THEOREM. Let H and K be locally bounded, predictable processes and let X be a semimartingale. Then $HK\cdot X = H\cdot(K\cdot X) = K\cdot(H\cdot X)$, provided one of the processes is real-valued.

Proof: The theorem is a standard result when all three processes are real-valued [11, p. 313]. In certain other cases, we may apply Theorem 6 and reduce to the real-valued case.

(a) Suppose H and K are real-valued and X is E-valued. Then for every f in E,

$$\begin{aligned}(f, HK\cdot X) &= HK\cdot(f, X) = H\cdot(K\cdot(f,X)) \\ &= H\cdot(f, K\cdot X) = (f, H\cdot(K\cdot X)),\end{aligned}$$

outside of an evanescent set. In the separable Hilbert space E, there exists a countable norming set (f_n) such that, if

$(f_n, x) = (f_n, y)$ for all n, then $x = y$. Hence, outside of the one evanescent set determined by all of the f_n, $HK \cdot X = H \cdot (K \cdot X)$. By symmetry, we have equality also for $K \cdot (H \cdot X)$.

(b) If H and X are real-valued and K is E-valued, then for every f in E,

$$(f,\ H \cdot (K \cdot X)) = H \cdot (f,\ K \cdot X) = H \cdot ((f, K) \cdot X)$$

$$= H(f,\ K) \cdot X = (f,\ HK) \cdot X = (f,\ HK \cdot X).$$

Also,

$$(f,\ H \cdot (K \cdot X)) = H \cdot ((f,\ K) \cdot X) = (f,\ K) \cdot (H \cdot X)$$

$$= (f,\ K \cdot (H \cdot X)).$$

Hence, outside of one evanescent set,

$$H \cdot (K \cdot X) = HK \cdot X = K \cdot (H \cdot X).$$

By symmetry, the result holds if K and X are real and H is E-valued.

(c) Suppose H and K are E-valued and X is real. By symmetry, it suffices to show $(H,\ K) \cdot X = H \cdot (K \cdot X)$. Suppose we have the result for bounded H and K. Then if H and K are locally bounded, both zero at 0, we let stopping times T_n increase to ∞ such that $H1_{[0,T_n]}$ and $K1_{[0,T_n]}$ are bounded for each n. Then,

$$
\begin{aligned}
(H, K)\cdot X &= \lim_{n\to\infty}(H, K)1_{[0,T_n]}\cdot X \\
&= \lim_{n\to\infty}(H1_{[0,T_n]}, K1_{[0,T_n]})\cdot X \\
&= \lim_{n\to\infty} H1_{[0,T_n]}\cdot(K1_{[0,T_n]}\cdot X) \\
&= \lim_{n\to\infty} H1_{[0,T_n]}\cdot(K\cdot X)^{T_n} \\
&= \lim_{n\to\infty} H1_{[0,T_n]}\cdot(K\cdot X) \\
&= H\cdot(K\cdot X).
\end{aligned}
$$

Then, in general,

$$
\begin{aligned}
(H, K)\cdot X &= (H_0, K_0)X_0 + (H, K)1_{]0,\infty[}\cdot X \\
&= (H_0, K_0X_0) + (H1_{]0,\infty[}, K1_{]0,\infty[})\cdot X \\
&= (H_0, (K\cdot X)_0) + H1_{]0,\infty[}\cdot(K1_{]0,\infty[}\cdot X).
\end{aligned}
$$

However, letting $L = H1_{]0,\infty[}$, we have

$$
\begin{aligned}
L\cdot(K\cdot X) &= L\cdot(K_0X_0 + K1_{]0,\infty[}\cdot X) \\
&= (L_0, K_0X_0) + L\cdot(K1_{]0,\infty[}\cdot X) \\
&= L\cdot(K1_{]0,\infty[}\cdot X).
\end{aligned}
$$

The last equality holds since $L_0 = 0$. We note that $B = K_0X_0$ is a process of finite variation; so,

$$(L\cdot B)_t = \int^t L_s dB_s = (L_0,\ K_0X_0).$$

We then have that

$$(H,\ K)\cdot X = (H_0,\ (K\cdot X)_0) + H1_{]0,\infty[}\cdot(K\cdot X)$$

$$= H\cdot(K\cdot X).$$

So it suffices to prove the result for bounded H and K. Let $X = X_0 + M + A$, where A is a process of finite variation, M is a locally square integrable local martingale, and $M_0 = 0 = A_0$. Then

$$(H,\ K)\cdot X = (H_0,\ K_0)X_0 + (H,\ K)\cdot M + (H,\ K)\cdot A$$

and

$$H\cdot(K\cdot X) = (H_0,\ (K\cdot X)_0) + H\cdot(K\cdot M) + H\cdot(K\cdot A).$$

By Theorem 5(c), $(K\cdot A)_t = \int^t K_s dA_s$ and thus

$$H\cdot(K\cdot A) = \int^t (H_s,\ K_s)dA_s = (H,\ K)\cdot A.$$

By Theorem 5(g), we have

$$(H_0,\ (K\cdot X)_0) = (H_0,\ K_0X_0) = (H_0,\ K_0)X_0.$$

To show the result for M, we assume first that M is a square integrable martingale and that H and K are elementary. To conform with previous notation, we write M = X. Let

$$H = L_{-1}1_{[0]} + L_0 1_{]t_0,t_1]} + \dots + L_n 1_{]t_n,t_{n+1}]}$$

and

$$K = J_{-1}1_{[0]} + J_0 1_{]s_0,s_1]} + \dots + J_m 1_{]s_m,s_{m+1}]}.$$

Then

$$(H,\ K) = (L_{-1},\ J_{-1})1_{[0]}$$

$$+ \sum_{j=0}^{m} \sum_{i=0}^{n} (L_i,\ J_j)(1_{]s_j \wedge t_{i+1},\, s_{j+1} \wedge t_{i+1}]} - 1_{]s_j \wedge t_i,\, s_{j+1} \wedge t_i]}),$$

$$(H,\ K)\cdot X = (L_{-1},\ J_{-1})X_0$$

$$+ \sum_{j=0}^{m} \sum_{i=0}^{n} (L_i,\ J_j)(X^{t_{i+1} \wedge s_{j+1}} - X^{t_{i+1} \wedge s_j})$$

$$- \sum_{j=0}^{m} \sum_{i=0}^{n} (L_i,\ J_j)(X^{t_i \wedge s_{j+1}} - X^{t_i \wedge s_j}),$$

$$H\cdot(K\cdot X) = (L_{-1},\ (K\cdot X)_0) + \sum_{i=0}^{n} (L_i,\ (K\cdot X)^{t_{i+1}} - (K\cdot X)^{t_i})$$

$$= (L_{-1},\ J_{-1}X_0) + \sum_{i=0}^{n} (L_i,\ K\cdot X^{t_{i+1}} - K\cdot X^{t_i})$$

$$= (L_{-1},\ J_{-1})X_0 + \sum_{i=0}^{n} (L_i,\ K\cdot X^{t_{i+1}}) - \sum_{i=0}^{n} (L_i,\ K\cdot X^{t_i}),$$

$$\sum_{i=0}^{n} (L_i,\ K\cdot X^{t_{i+1}}) = \sum_{i=0}^{n} (L_i,\ J_{-1}X_0)$$

$$+ \sum_{i=0}^{n} \sum_{j=0}^{m} (L_i,\ J_j)(X^{t_{i+1} \wedge s_{j+1}} - X^{t_{i+1} \wedge s_j}), \quad \text{and}$$

$$\sum_{i=0}^{n} (L_i,\ K\cdot X^{t_i}) = \sum_{i=0}^{n} (L_i,\ J_{-1}X_0)$$

$$+ \sum_{i=0}^{n} \sum_{j=0}^{m} (L_i,\ J_j)(X^{t_i \wedge s_{j+1}} - X^{t_i \wedge s_j}).$$

Hence, $(H, K)\cdot X = H\cdot(K\cdot X)$ for elementary H and K.

If K is bounded and H is elementary, let (K^n) be a sequence of elementary processes converging to K in $L_p^2(X)$. Since H is also bounded and

$$E[\int^\infty |(H_s, K_s^n) - (H_s, K_s)|^2 d\langle X , X\rangle_s]$$

$$\leq E[\int^\infty |H_s|^2|K_s^n - K_s|^2 d\langle X, X\rangle_s],$$

we have that (H, K^n) converges to (H, K) in $L_p^2(X)$. Thus, by Theorem 2(a), $(H, K^n)\cdot X$ converges to $(H, K)\cdot X$ in M^2.

$K^n\cdot X$ converges to $K\cdot X$ in M_E^2, also by Theorem 2(a), and since H is elementary, $H\cdot(K^n\cdot X)$ converges to $H\cdot(K\cdot X)$ in M^2. For if H is as above, then

$$H\cdot(K^n\cdot X) = (L_{-1}, (K^n\cdot X)_0) + \sum_{i=0}^{n} (L_i, (K^n\cdot X)^{t_{i+1}} - (K^n\cdot X)^{t_i})$$

which, since each L_i is bounded, converges in M^2 to

$$(L_{-1}, (K\cdot X)_0) + \sum_{i=0}^{n} (L_i, (K\cdot X)^{t_{i+1}} - (K\cdot X)^{t_i}) = H\cdot(K\cdot X).$$

Then since $(H, K^n)\cdot X = H\cdot(K^n\cdot X)$ for each n, we have $(H, K)\cdot X = H\cdot(K\cdot X)$ for H elementary and K bounded.

If both H and K are bounded, we let (H^n) be a sequence of elementary processes converging to H in $L_p^2(X)$. As above, since

K is bounded, (H^n, K) converges to (H, K) in $L_p^2(X)$; hence, $(H^n, K)\cdot X$ converges to $(H, K)\cdot X$ in M^2.

By Theorem 1.7, we have for $L^n = H^n - H$ that

$$E[\int^\infty |L_s^n|^2 d\langle K\cdot X, K\cdot X\rangle_s] = E[\int^\infty |L_s^n|^2 |K_s|^2 d\langle X, X\rangle_s]$$

and hence H^n converges to H in $L_p^2(K\cdot X)$. Thus, $H^n\cdot(K\cdot X)$ converges to $H\cdot(K\cdot X)$ in M^2. Since then $(H^n, K)\cdot X = H^n\cdot(K\cdot X)$ for each n, we obtain $(H, K)\cdot X = H\cdot(K\cdot X)$, for bounded H and K and a square integrable martingale X.

If X is locally square integrable, we let stopping times T_n increase to ∞ such that X^{T_n} is square integrable. Then

$$(H\cdot(K\cdot X))^{T_n} = H\cdot(K\cdot X)^{T_n} = H\cdot(K\cdot X^{T_n}) = (H, K)\cdot X^{T_n}.$$

Taking the limit as $n\to\infty$, we obtain

$$H\cdot(K\cdot X) = (H, K)\cdot X.$$

(d) Suppose X is E-valued, H is E-valued, and K is real-valued. Applying Theorem 1.7 and interchanging the inner product by scalar product when necessary, we prove, as in (c), that $HK\cdot X = H\cdot(K\cdot X)$. Using this lone result, we shall obtain in Chapter V a more general result in the case when H is an optional process. From this general result, we shall then obtain equality for $K\cdot(H\cdot X)$.

8 THEOREM. Let (H^k) be a uniformly bounded sequence of predictable processes which converges pointwise to zero. Then for any

semimartingale X, $(H^K \cdot X)^*_t$ converges in probability to zero for every t.

9 REMARKS. Let X be a square integrable martingale, real or E-valued. Using the geometry of the Hilbert space M^2, we project X onto the closed subspace $(M^2)^c$ of continuous square integrable martingales which are zero at 0. The orthogonal complement $(M^2)^d$ of $(M^2)^c$ is called the space of purely discontinuous square integrable martingales. Then X has a unique decomposition $X = X^c + X^d$, where X^c is in $(M^2)^c$ and X^d is in $(M^2)^d$.

Now if H is in $L^2_p(X)$, $H \cdot X$ is in M^2. It can be shown that

$$(H \cdot X)^c = H \cdot X^c \tag{9.1}$$

and

$$(H \cdot X)^d = H \cdot X^d. \tag{9.2}$$

This concept extends to local martingales as well. In Chapters III and V, we shall study this concept using the optional stochastic integral, of which the predictable stochastic integral is a special case. We shall then obtain results analogous to (9.1) and (9.2) for optional H, thus recovering (9.1) and (9.2) for predictable H.

3. The Square Bracket

In this section, we define and give basic properties of the square bracket $[X, Y]$ of two E-valued semimartingales X and Y. The classical results appear in Déllacherie and Meyer [4].

1 DEFINITION. Let X and Y be E-valued semimartingales. We define the square bracket of X and Y to be the real, adapted, cadlag process given by

$$[X, Y]_t = (X_t, Y_t) - (X_- \cdot Y)_t - (Y_- \cdot X)_t.$$

2 THEOREM. Let X and Y be semimartingales.

(a) The mapping $\{X, Y\} \to [X, Y]$ is bilinear.

(b) $[X, Y] = [Y, X]$.

(c) $[aX, Y] = a[X, Y]$, for every scalar a.

(d) $\Delta[X, Y] = (\Delta X, \Delta Y)$; $[X, Y]_0 = (X_0, Y_0)$.

3 THEOREM. For all semimartingales X and Y, [X, Y] is a process of finite variation. Moreover, [X, X] is an increasing process.

4 THEOREM. If one of X and Y is a process of finite variation, then $[X, Y]_t = \sum_{s \leq t} (\Delta X_s, \Delta Y_s)$. Moreover, if one of X and Y is a predictable process of finite variation and the other is a local martingale, then [X, Y] is also a local martingale.

5 THEOREM. Let X and Y be local martingales. Then [X, Y] is the unique process of finite variation such that the following properties hold:

(1) $(X, Y) - [X, Y]$ is a local martingale which is zero at 0.

(2) $\Delta[X, Y] = (\Delta X, \Delta Y)$.

If X and Y are both square integrable martingales, then $(X, Y) - [X, Y]$ is a uniformly integrable martingale.

6 COROLLARY. If X and Y are square integrable martingales, then for every stopping time T, $[X, Y]_T$ is integrable and $E[[X, Y]_T] = E[(X_T, Y_T)]$.

Conversely, suppose X is a local martingale and $[X, X]_\infty$ is integrable. We aim to show that X is in fact a square integrable martingale. We first reduce to the case when X is zero at 0. Since X_0 is a process of finite variation, by Theorem 4 we have that $[X, X_0] = (X_0, X_0)$ and $[X_0, X_0] = (X_0, X_0)$. Then, by Theorem 2, we have $[X - X_0, X - X_0] = [X, X] - 2[X, X_0] + [X_0, X_0] =$ $= [X, X] - (X_0, X_0)$. Hence, $[X - X_0, X - X_0]_\infty$ is also integrable, since it is dominated by $[X, X]_\infty$. Since $|X_0|^2 = (X_0, X_0) =$ $[X, X]_0 \leq [X, X]_\infty$, X_0 is square integrable; thus, if we can show $X - X_0$ is a square integrable martingale, then X will be also.

7 THEOREM. If X is a local martingale and $[X, X]_\infty$ is integrable, then X is a square integrable martingale.

Proof: By the above remarks, we may assume $X_0 = 0$. Let $Y = |X|^2 - [X, X]$. Then Y and X are both local martingales which are zero at 0; hence, there exists a sequence of stopping times (T_n) which increase to infinity such that Y^{T_n} and X^{T_n} are uniformly integrable martingales for each n. Then for every stopping time T, we have

$$X_{T \wedge T_n} = X_T^{T_n} = E(X_{T_n} | F_T). \tag{7.1}$$

Since $E[Y_{T_n}] = E[Y_0] = 0$ and $E[[X, X]_{T_n}] < \infty$, we have, by (7.1),

$$E[|X_{T \wedge T_n}|^2] \leq E[|X_{T_n}|^2] = E[[X, X]_{T_n}] \tag{7.2}$$

$$\leq E[[X, X]_\infty].$$

By Fatou's lemma, we have

$$E[|X_T|^2 1_{(T<\infty)}] \leq \liminf_{n\to\infty} E[|X_{T \wedge T_n}|^2] \tag{7.3}$$

$$\leq E[[X, X]_\infty]$$

$$< \infty.$$

Hence, the set $C = \{|X_T| 1_{(T<\infty)}: T \text{ is a stopping time}\}$ is bounded in L^2. It remains to show that X is a true martingale.

The set C is bounded in the reflexive space $L^2(\Omega, F, P)$; thus, C is weakly sequentially compact in L^2 [7, II.3.28] and also in L^1, since the dual of L^1 is contained in the dual of L^2. We then have [7, IV.8.11]

$$\lim_{P(A)\to 0} \int_A |X_T| 1_{(T<\infty)} dP = 0, \text{ uniformly in } T. \tag{7.4}$$

Let $s \leq t$ and let A in F_s. Then for every n

$$\int_A X_{t \wedge T_n} dP = \int_A X_{s \wedge T_n} dP. \tag{7.5}$$

Since (T_n) increases to infinity, $\lim_{n\to\infty} X_{t\wedge T_n} = X_t$ almost surely. By (7.4) and the Vitali Convergence Theorem, we have that $X_{t\wedge T_n}$ converges to X_t in L^1. Likewise, $X_{s\wedge T_n}$ converges to X_s in L^1. Taking the limit in (7.5) as n tends to infinity, we obtain

$$\int_A X_t dP = \int_A X_s dP. \tag{7.6}$$

That is, $E(X_t | F_s) = X_s$ a.s. Hence, X is a square integrable martingale.

8 THEOREM. Let X and Y be semimartingales.

(a) $[X^T, Y] = [X, Y]^T$, for every stopping time T.

(b) $[HX, Y] = H[Y, X]$, for every real, F_0-measurable, random variable H.

(c) $[X^{T-}, Y] = [X, Y]^{T-}$, for every stopping time T.

9 THEOREM. Let X be an E-valued semimartingale and let Y be a real-valued semimartingale. Then for every f in E, $[X, fY] = [(X, f), Y]$.

Proof: We first assume that X and Y are local martingales. Then $(X, fY) - [X, fY]$ is a local martingale which is zero at 0 and $\Delta[X, fY] = (\Delta X, \Delta(fY))$. However, $(X, fY) = (X, f)Y$ and $(\Delta X, \Delta(fY)) = (\Delta X, f)\Delta Y = \Delta(X, f)\Delta Y$. Hence, $(X, f)Y - [X, fY]$ is a local martingale which is zero at 0 and $\Delta[X, fY] = \Delta(X, f)\Delta Y$. By the uniqueness in Theorem 5, which applies in particular to real local martingales, $[(X, f), Y] = [X, fY]$.

If one of X and Y is a process of finite variation, then

$$[X, fY]_t = \sum_{s\le t} (\Delta X_s, \Delta(fY)_s) = \sum_{s\le t} \Delta(X, f)_s \Delta Y_s = [(X, f), Y]_t.$$

If X and Y are semimartingales, write $X = M + A$ and $Y = N + B$, where M and N are local martingales and A and B are processes of finite variation. Then

$$\begin{aligned}[X, fY] &= [M, fN] + [M, fB] + [A, fN] + [A, fB] \\ &= [(M, f), N] + [(M, f), B] + [(A, f), N] + [(A, f), B] \\ &= [(X, f), Y].\end{aligned}$$

10 THEOREM. Suppose X takes values in a closed subspace E_0 of E. If Y takes values in the orthogonal complement of E_0, then $[X, Y] = 0$.

Proof: Since $(X, Y) = 0$, we have that $[X, Y] = X_-\cdot Y + Y_-\cdot X$. Thus, it suffices to show that if H is a locally bounded, predictable process with values in $E_0^\perp$, then $H\cdot X = 0$. This argument applies as well to $X_-\cdot Y$, since $E_0^\perp$ is also closed and $E_0 = (E_0^\perp)^\perp$.

If X is a square integrable martingale and H is elementary, then $H\cdot X = 0$ by Theorem 1.5(d). If H is in $L_p^2(X)$, $H\cdot X$ is the limit in M^2 of $(H^n\cdot X)$, where (H^n) is any sequence of elementary processes converging to H in $L_p^2(X)$; hence, $H\cdot X = 0$.

If X is a semimartingale and H is bounded, we may decompose X as $X = X_0 + M + A$, where M is locally square integrable, A is of finite variation, $M_0 = 0 = A_0$, and M and A still take values in E_0. By localizing and stopping, we have $H\cdot M = 0$. Moreover, $(H\cdot A)_t = \int^t H_s dA_s = 0$, since $(H, A) = 0$. Hence, $H\cdot X = (H_0, X_0) + H\cdot M + H\cdot A$

= 0. By localizing and stopping, we obtain the result for locally bounded H. Letting $H = Y_-$, we obtain $[X, Y] = 0$.

4. Applications of the Square Bracket

In this section, we list some analytic properties of the square bracket and give two applications of the square bracket to the predictable stochastic integral. We start with the famous Kunita-Watanabe inequality, which holds also for the sharp bracket, provided the sharp bracket exists.

1 THEOREM. Let X and Y be E-valued semimartingales and let H and K be real, bounded, measurable processes. Then almost surely,

$$\int^\infty |H_s||K_s||d[X, Y]_s| \leq \left(\int^\infty H_s^2 d[X, X]_s\right)^{1/2}\left(\int^\infty K_s^2 d[Y, Y]_s\right)^{1/2}.$$

2 COROLLARY.

(a) $[X+Y, X+Y]_t^{1/2} \leq [X, X]_t^{1/2} + [Y, Y]_t^{1/2}$.

(b) $[X+Y, X+Y]_t \leq 2[X, X]_t + 2[Y, Y]_t$.

Proof:

(a) By Theorem 1, we have that, outside of an evanescent set,

$$\begin{aligned} |[X, Y]_t| &= \left|\int^t d[X, Y]_s\right| \leq \int^t |d[X, Y]_s| \\ &\leq \left(\int^t d[X, X]_s\right)^{1/2}\left(\int^t d[Y, Y]_s\right)^{1/2} \\ &= [X, X]_t^{1/2}[Y, Y]_t^{1/2}. \end{aligned}$$

Thus,

$$\begin{aligned}[X+Y, X+Y]_t &= [X, X]_t + 2[X, Y]_t + [Y, Y]_t \\ &\le [X, X]_t + 2[X, X]_t^{1/2}[Y, Y]_t^{1/2} + [Y, Y]_t \\ &= ([X, X]_t^{1/2} + [Y, Y]_t^{1/2})^2.\end{aligned}$$

By taking square roots, we obtain the result.

(b) We use the properties obtained in Theorem 3.2. We have

$$[X, Y] = [X, X] + [X, Y-X] \quad \text{and}$$

$$\begin{aligned}[Y, X] &= [Y, Y] + [Y, X-Y] \\ &= [Y, Y] + (-1)^2[Y, X-Y] \\ &= [Y, Y] + [-Y, Y-X].\end{aligned}$$

Thus,

$$\begin{aligned}[X+Y, X+Y] &= [X, X] + [Y, Y] + [X, Y] + [Y, X] \\ &= 2[X, X] + 2[Y, Y] + [X-Y, Y-X] \\ &= 2[X, X] + 2[Y, Y] - [Y-X, Y-X] \\ &\le 2[X, X] + 2[Y, Y],\end{aligned}$$

since $[Y-X, Y-X]$ is, by Theorem 3.3, an increasing process and hence is nonnegative.

3 THEOREM. Let X be a square integrable martingale. Then $\langle X, X\rangle$ is the predictable dual projection of $[X, X]$.

Proof: By Theorem 3.6, $E[[X, X]_\infty] = E[|X_\infty|^2] < \infty$; thus, $[X, X]$ is an integrable increasing process and $[X, X]^p$ exists. $|X|^2 - [X, X]$ and $|X|^2 - \langle X, X\rangle$ are uniformly integrable martingales which are zero at 0; thus, $[X, X] - \langle X, X\rangle$ is also a uniformly integrable martingale which is zero at 0. Since $\langle X, X\rangle$ is predictable, $\langle X, X\rangle = [X, X]^p$.

4 THEOREM. Let A be an E-valued process of locally integrable variation and let A^p be its predictable compensator. Then $E[[A^p, A^p]_\infty] \leq E[[A, A]_\infty]$.

Proof: $M = A - A^p$ is a local martingale which is zero at 0. Let (T_n) be a sequence of stopping times which increase to infinity such that M^{T_n} is a uniformly integrable martingale and A^{T_n} is of integrable variation. Then $M^{T_n} = A^{T_n} - (A^p)^{T_n}$ is a uniformly integrable martingale which is zero at 0; hence $(A^{T_n})^p = (A^p)^{T_n}$. By monotone convergence, $E[[A^{T_n}, A^{T_n}]_\infty] = E[[A, A]_{T_n}]$ converges to $E[[A, A]_\infty]$ and $E[[(A^{T_n})^p, (A^{T_n})^p]_\infty] = E[[A^p, A^p]_{T_n}]$ converges to $E[[A^p, A^p]_\infty]$, as n goes to infinity. So we may assume A, and hence A^p, is of integrable variation.

Since $A_0 - A^p_0 = M_0 = 0$, we have that $(A-A_0) - (A^p-A^p_0) = A - A^p = M$; thus, $(A-A_0)^p = A^p - A^p_0$. We may now make two more assumptions on A. First, we may assume $E[[A, A]_\infty] < \infty$, for otherwise the theorem is true. Then $E[|A_0|^2] = E[[A, A]_0] \leq E[[A, A]_\infty] < \infty$. Secondly, since $[(A-A_0)^p, (A-A_0)^p] = [A^p-A_0^p, A^p-A_0^p] = [A^p, A^p] - |A_0^p|^2 = [A^p, A^p] - |A_0|^2$ and $[A-A_0, A-A_0] = [A, A] - |A_0|^2$, if we can show the result for $A - A_0$, we will then have the result for A by adding $E[|A_0|^2]$. So we may assume $A_0 = 0 = A_0^p$. Henceforth, we shall write $B = A^p$.

Since A and B are of integrable variation and zero at 0, the random vectors A_T, B_T, and hence M_T, are integrable for every stopping time T. In particular, for every predictable time T

$$\begin{aligned} E(A_T|F_{T-}) &= E(B_T|F_{T-}) + E(M_T|F_{T-}) \\ &= B_T + M_{T-} \quad \text{a.s.} \end{aligned}$$

That is, the predictable projection of A exists and ${}^pA = B + M_-$. Then ${}^pA - A_- = B - B_- = \Delta B$.

For every f in E, the set $\{(f, \Delta B) \neq 0\} = \{\Delta(f, B) \neq 0\}$ is contained in a countable union of graphs of predictable times [4, p. xiii]. Letting f range through a countable norming set, we have that the set $\{\Delta B \neq 0\}$ is also contained in a countable union of graphs of predictable times. Since this set is predictable, it is in fact indistinguishable from a countable union of disjoint graphs of predictable times (T_n) [3, IV.89]. We then have that

$$E[[B, B]_\infty] = E[\sum_s |\Delta B_s|^2]$$

$$= E[\sum_i |\Delta B_{T_i}|^2]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} E[|\Delta B_{T_i}|^2].$$

Since $\sum_{i=1}^{n} |\Delta A_{T_i}|^2 = \sum_{i=1}^{n} \Delta[A, A]_{T_i} \leq [A, A]_\infty$, it suffices to show $E[|\Delta B_{T_i}|^2] \leq E[|\Delta A_{T_i}|^2]$ for each i. Let $T = T_i$. Since T is predictable, we have a.s.

$$\Delta B_T = {}^pA_T - A_{T_-} = E(A_T|F_{T_-}) - A_{T_-}$$

$$= E(\Delta A_T|F_{T_-}).$$

Hence, $|\Delta B_T|^2 \leq E(|\Delta A_T|^2|F_{T_-})$ a.s. and $E[|\Delta B_T|^2] \leq E[|\Delta A_T|^2]$. Thus,

$$E[[B, B]_\infty] \leq \lim_{n\to\infty} \sum_{i=1}^{n} E[|\Delta A_{T_i}|^2]$$

$$= \lim_{n\to\infty} E[\sum_{i=1}^{n} |\Delta A_{T_i}|^2]$$

$$\leq E[[A, A]_\infty].$$

5 THEOREM. If X is a special semimartingale, then $[X, X]_t^{1/2}$ is locally integrable.

Proof: Since $[X, X]_t^{1/2}$ is not necessarily zero at 0, we must show that the process $[X, X]_t^{1/2} - [X, X]_0^{1/2}$ is locally integrable.

However,

$$[X, X]_t^{1/2} - [X, X]_0^{1/2} \leq ([X, X]_t - [X, X]_0)^{1/2}$$
$$= [X-X_0, X-X_0]_t^{1/2} .$$

So it suffices to prove the result for $X - X_0$. We may then assume that X, and hence [X, X], is zero at 0.

We may then decompose X as $X = M + A$, where M is a local martingale, A is a predictable process of finite variation, and $M_0 = 0 = A_0$. By Theorem 2(a),

$$[X, X]_t^{1/2} \leq [M, M]_t^{1/2} + [A, A]_t^{1/2} .$$

Since A is of finite variation,

$$[A, A]_t^{1/2} = (\sum_{s\leq t} |\Delta A_s|^2)^{1/2}$$
$$\leq \sum_{s\leq t} |\Delta A_s|$$
$$\leq \int^t |dA_s| .$$

But $\int^t |dA_s|$ is a real, predictable process of finite variation and so is locally integrable [4, VI.80]. Thus, $[A, A]_t^{1/2}$ is also locally integrable.

Let $T_n = \inf\{t: n \leq |M_t|\}$. Since M is cadlag, the stopping times T_n increase to infinity. Let (R_n) be a sequence of stopping times which increase to infinity such that M^{R_n} is a uniformly integrable martingale. Then let $S_n = T_n \wedge R_n$. Then $M_{S_n} = M^{R_n}_{S_n}$ is integrable for each n.

Let $M^*_t = \sup_{s \le t} |M_s|$. Then

$$M^*_{S_n} = \sup_{s \le S_n} |M_s| \le \sup_{s < S_n} |M_s| \vee |M_{S_n}|$$

$$\le n + |M_{S_n}|.$$

Thus, $M^*_{S_n}$ is integrable and M^* is locally integrable.

Since $[M, M]^{1/2}_{t-}$ is a left continuous, increasing process, it is also locally integrable. Then

$$\begin{aligned} [M, M]^{1/2}_t - [M, M]^{1/2}_{t-} &\le (\Delta[M, M]_t)^{1/2} \\ &= |\Delta M_t| \\ &\le 2M^*_t. \end{aligned}$$

Hence, $[M, M]^{1/2}_t \le [M, M]^{1/2}_{t-} + 2M^*_t$. Thus, $[M, M]^{1/2}_t$, and hence $[X, X]^{1/2}_t$, is locally integrable.

The proof of the following theorem is the same as in the real-valued case [4, VIII.22 and 30].

6 THEOREM. Let X and Y be E-valued semimartingales and let H be a real, locally bounded, predictable process. Then $[H \cdot X, Y] = H \cdot [X, Y]$. The same result is true if X and Y are square integrable martingales and H is in $L^2_p(X)$.

7 THEOREM. If X is an E-valued semimartingale and H is a real, locally bounded, predictable process, or if X is a real semimartingale and H is an E-valued, locally bounded, predictable process, then $[H\cdot X, H\cdot X] = |H|^2\cdot[X, X]$. When X is a square integrable martingale, then result holds for H in $L_p^2(X)$.

Proof: We write $X = M + A$, where M is a locally square integrable local martingale which is zero at 0 and A is a process of finite variation. Then

$$[H\cdot X, H\cdot X] = [H\cdot M, H\cdot M] + 2[H\cdot M, H\cdot A] + [H\cdot A, H\cdot A]. \tag{7.1}$$

Since $H\cdot A$ is also of finite variation, we have by Theorem 3.4 (using juxtaposition for product and inner product)

$$\begin{aligned}[H\cdot M, H\cdot A]_t &= \sum_{s\leq t} \Delta(H\cdot M)_s\Delta(H\cdot A)_s \\ &= \sum_{s\leq t} H_s\Delta M_s H_s\Delta A_s \\ &= \sum_{s\leq t} |H_s|^2\Delta[M, A]_s \\ &= \int^t |H_s|^2 d[M, A]_s.\end{aligned} \tag{7.2}$$

Likewise,

$$[H\cdot A, H\cdot A]_t = \int^t |H_s|^2 d[A, A]_s. \tag{7.3}$$

To prove the result for $[H\cdot M, H\cdot M]$, we may assume, by localizing and stopping, that M is square integrable. We then let H be in

$L_0^2(M)$. Since

$$\begin{aligned}\Delta[H\cdot M, H\cdot M] &= (\Delta(H\cdot M))^2\\ &= (H\Delta M)^2\\ &= |H|^2\Delta[M, M]\\ &= \Delta(|H|^2\cdot[M, M]),\end{aligned}$$

and in view of Theorem 3.5, which characterizes the square bracket for local martingales, it suffices to show

$$B = |H\cdot M|^2 - |H|^2\cdot[M, M]$$

is a local martingale which is zero at 0.

By Theorem 1.7, we have $\langle H\cdot M, H\cdot M\rangle = |H|^2\cdot\langle M, M\rangle$, and since $H\cdot M$ is a square integrable martingale, we have

$$|H\cdot M|^2 - \langle H\cdot M, H\cdot M\rangle = |H\cdot M|^2 - |H|^2\cdot\langle M, M\rangle \tag{7.4}$$

is a uniformly integrable martingale which is zero at 0.

Since H is predictable, the predictable dual projection of $|H|^2\cdot[M, M]$ is $|H|^2\cdot\langle M, M\rangle$ and hence

$$|H|^2\cdot[M, M] - |H|^2\cdot\langle M, M\rangle \tag{7.5}$$

is a uniformly integrable martingale which is zero at 0. Subtracting (7.4) from (7.3) gives us that $|H\cdot M|^2 - |H|^2\cdot[M, M]$ is also a uniformly integrable martingale, which is zero at 0. Hence,

$$[H\cdot M,\ H\cdot M] = |H|^2\cdot[M,\ M]. \tag{7.6}$$

Since $|H|^2\cdot[M, M] + 2|H|^2\cdot[M, A] + |H|^2\cdot[A, A] = |H|^2\cdot[X, X]$, the result follows by comparing (7.2), (7.3), and (7.6) with (7.1).

Let X and Y be semimartingales. Write $X = M + A$ and $Y = N + B$, where M and N are locally square integrable local martingales which are zero at 0, and A and B are processes of finite variation. Then

$$[X,\ Y] = [M,\ N] + [M,\ B] + [A,\ N] + [A,\ B].$$

The last three summands are uniquely expressed by Theorem 3.4. We may define [M, N] by pasting together, for if (T_n) is a sequence of stopping times which increase to infinity and which reduces both M and N, then

$$[M,\ N]^{T_n} = [M^{T_n},\ N^{T_n}].$$

So it is really a matter of defining the square bracket for square integrable martingales. An alternate approach for defining the square bracket in this case, one which does not use the stochastic integral, is as follows: If X and Y are in M_E^2, let

$$[X,\ Y] = \langle X^c,\ Y^c\rangle + \sum_{s\le t} (\Delta X_s,\ \Delta Y_s). \tag{8.1}$$

This definition is actually a consequence of Theorem 3.3. Since [X, Y] is a process of finite variation, it may be decomposed uniquely as

$$(8.2)\qquad [X, Y]_t = [X, Y]_t^c + \sum_{s\le t} \Delta[X, Y]_s$$

$$= [X, Y]_t^c + \sum_{s\le t} (\Delta X_s, \Delta Y_s).$$

We know $\langle X^c, Y^c\rangle$ exists, since X^c and Y^c are in M_E^2, and since $[X, Y]^c$ is continuous, and hence predictable, we will have $[X, Y]^c = \langle X^c, Y^c\rangle$.

Using (8.1), Ustanel [15, II.5] showed that if H is E-valued, locally bounded, and predictable, Y is a real local martingale, and X is an E-valued local martingale, then

$$(8.3)\qquad [Y, H\cdot X] = [H\cdot Y, X].$$

The result extends immediately for semimartingales X and Y, for if one of X and Y is of finite variation, then

$$[Y, H\cdot X]_t = \sum_{s\le t} \Delta Y_s \Delta(H\cdot X)_s$$

$$= \sum_{s\le t} \Delta Y_s (H_s, \Delta X_s)$$

$$= \sum_{s\le t} (H_s \Delta Y_s, \Delta X_s)$$

$$= \sum_{s\le t} (\Delta(H\cdot Y)_s, \Delta X_s)$$

$$= [H\cdot Y, X]_t.$$

Then decomposing semimartingales X and Y in the usual manner, $X = M + A$ and $Y = N + B$, we obtain

$$\begin{aligned}[Y, H\cdot X] &= [N, H\cdot M] + [B, H\cdot M] + [N, H\cdot A] + [B, H\cdot A] \\ &= [H\cdot N, M] + [H\cdot B, M] + [H\cdot N, A] + [H\cdot B, A] \\ &= [H\cdot Y, X].\end{aligned}$$

We state this result as the following:

8 THEOREM. If H is a locally bounded, predictable, E-valued process, Y is a real semimartingale, and X is an E-valued semimartingale, then

$$[Y, H\cdot X] = [H\cdot Y, X].$$

We conclude this section with one final theorem on the square bracket.

9 THEOREM. Let X be an E-valued square integrable martingale. Then for every f in E and for every real measurable process H we have

(a) $[(f, X), (f, X)] \leq |f|^2[X, X]$ and

(b) $\int^\infty |H_s| d[(f, X), (f, X)]_s \leq |f|^2 \int^\infty |H_s| d[X, X]_s$.

Proof:

(a) We note that (f, X) is a real square integrable martingale. Then for every stopping time T,

$$\begin{aligned}E[[(f, X), (f, X)]_T] &= E[|(f, X_T)|^2] \\ &\leq |f|^2 E[|X_T|^2] \\ &= |f|^2 E[[X, X]_T].\end{aligned}$$

Hence, by the cross-section theorem,

$$[(f, X), (f, X)] \leq |f|^2[X, X].$$

(b) By truncation and monotone convergence, it suffices to prove the result for bounded, measurable H. We shall use a variant of the monotone class theorem [3, I.22 (22.3)].

Let A be the set of processes which satisfy the theorem. Then A is closed under bounded monotone convergence. That is, if (H^n) is a uniformly bounded increasing sequence of positive processes in A, then $H = \lim_n H^n$ is in A, by the monotone convergence theorem.

Let C be the infimum-closed vector space of processes of the form

$$H = K_{-1}1_{[0]} + K_0 1_{]t_0,t_1]} + \cdots + K_n 1_{]t_n,t_{n+1}]} \tag{9.1}$$

where K_i are bounded and measurable (not necessarily adapted) and $0 \leq t_0 \leq t_1 \leq \ldots \leq t_{n+1} \leq \infty$.

We note that C also contains 1. It suffices to show C is contained in A.

If H has the form in (9.1), then

$$|H| = |K_{-1}|1_{[0]} + |K_0|1_{]t_0,t_1]} + \cdots + |K_n|1_{]t_n,t_{n+1}]}.$$

We then have

$$\int^{\infty}|H_s|d[(f,X),(f,X)]_s$$

$$= |K_{-1}|[(f,X),(f,X)]_0 + \sum_{i=1}^{n}|K_i|[(f,X),(f,X)]_{t_i}^{t_{i+1}}$$

and

$$|f|^2\int^{\infty}|H_s|d[X,X]_s$$

$$= |f|^2|K_{-1}|[X,X]_0 + \sum_{i=1}^{n}|f|^2|K_i|[X,X]_{t_i}^{t_{i+1}}.$$

We have by (a) that

$$|K_{-1}|[(f,X),(f,X)]_0 \le |f|^2|K_{-1}|[X,X]_0;$$

hence, it suffices to show for each $i=1,\ldots,n$ that

$$[(f,X),(f,X)]_{t_i}^{t_{i+1}} \le |f|^2[X,X]_{t_i}^{t_{i+1}}.$$

To simplify the notation, we let $Y = (f, X)$ and we let $s = t_i$ and $t = t_{i+1}$, so that $s \le t$. Then,

$$\begin{aligned}[Y,Y]_s^t &= [Y,Y]_t - [Y,Y]_s \\ &= [Y^t-Y^s, Y^t-Y^s]_{\infty} \\ &= [(f, X^t-X^s), (f, X^t-X^s)]_{\infty}.\end{aligned}$$

We then apply part (a) to the square integrable martingale

$X^t - X^s$ to obtain

$$[Y, Y]_s^t \le |f|^2[X^t-X^s, X^t-X^s]_\infty$$

$$= |f|^2[X, X]_s^t$$

and the theorem is proved.

5. H^p and BMO^p Spaces

In this last section, we define quadratic H^p spaces and BMO^p spaces of E-valued martingales. Moreover, we shall show that the dual of H^1 is BMO^2. Once again, these results are classical and the definitions and proofs are nearly identical to the real-valued case. It is also possible to show the equivalence of all the BMO^p spaces, for $p \ge 1$, using the classical proofs. Rather than further weigh down this chapter, we will leave this last result, and its consequence that H^1 is complete, for the Appendix.

1 DEFINITION. We denote by H^p the space of E-valued local martingales X such that

$$\|X\|_{H^p} = \|[X, X]_\infty^{1/2}\|_{L^p} < \infty \text{ (for } p \le 1).$$

We note a few immediate results.

(a) By Theorem 3.2 (b) and (c) we have, for every scalar a, $[aX, aX] = a^2[X, X]$. Hence, $\|aX\|_{H^p} = |a|\|X\|_{H^p}$.

(b) By Theorem 4.2 (a), we have $[X+Y, X+Y]_\infty^{1/2} \leq [X, X]_\infty^{1/2} + [Y, Y]_\infty^{1/2}$. Thus, by Minkowski's inequality, $\|X + Y\|_{H^p} \leq \|X\|_{H^p} + \|Y\|_{H^p}$.

(c) If $\|X\|_{H^p} = 0$, then $[X, X]_\infty^{1/2} = 0$ a.s. Thus, $[X, X]_\infty$ is integrable, being almost surely null, and X is a square integrable martingale by Theorem 3.7. Moreover, $E[|X_\infty|^2] = E[[X, X]_\infty] = 0$. Hence, $X_\infty = 0$ a.s. and X is indistinguishable from 0, being right closed by X_∞.

Properties (a), (b), and (c) mean that, by identifying indistinguishable processes, H^p is a normed linear space under $\|\cdot\|_{H^p}$.

(d) It is immediate, since the property holds for L^p spaces, that if $1 \leq p_1 \leq p_2$, then $H^{p_2} \subseteq H^{p_1}$ and $\|X\|_{H^{p_1}} \leq \|X\|_{H^{p_2}}$.

(e) We consider the case of p = 2. If X is a local martingale and $\|X\|_{H^2} < \infty$, then $[X, X]_\infty$ is integrable. Hence, X is a square integrable martingale by Theorem 3.7. Then by Theorem 3.6,

$$\|X\|_{M^2}^2 = E[|X_\infty|^2] = E[(X_\infty, X_\infty)] = E[[X, X]_\infty] = \|X\|_{H^2}^2.$$

We then see that the spaces M_E^2, L^2, and H^2 are all isometrically isomorphic.

(f) The H^p space we have defined is classically called "quadratic" H^p space and is often denoted by H_2^p, in order to distinguish it from the "maximal" H^p space whose norm is given by $m_p(X) = \|X^*\|_{L^p}$.

2 THEOREM. Let X be an E-valued local martingale having paths of finite variation. Then $\|X\|_{H^1} \leq E[\int^\infty |dX_s|]$.

Proof: We have

$$\begin{aligned}([X, X]_\infty)^{1/2} &= (\sum_s |\Delta X_s|^2)^{1/2} \\ &\leq \sum_s |\Delta X_s| \\ &\leq \int^\infty |dX_s|.\end{aligned}$$

The theorem follows by taking expectations.

3 THEOREM. Bounded martingales are dense in H^1.

Proof: Let X be in H^1. Then the process stopped at 0, $X^0 = X_0$, is also in H^1 since

$$\begin{aligned}\|X^0\|_{H^1} &= E[[X^0, X^0]_\infty^{1/2}] \\ &= E[[X, X]_0^{1/2}] \\ &\leq E[[X, X]_\infty^{1/2}] \\ &= \|X\|_{H^1}.\end{aligned}$$

Thus, $X - X_0$ is also in H^1 by 1 (b). It then suffices to find bounded martingales (X^n) which converge to $X - X_0$ in H^1 and bounded martingales (Y^n) which converge to X^0 in H^1. In the latter case,

$$\|X^0\|_{H^1} = E[[X, X]_0^{1/2}] = E[|X_0|] = \|X_0\|_{L^1}. \tag{3.1}$$

Hence, X_0 is in $L^1(F_0)$ and there exist bounded F_0-measurable random variables (f_n) which converge to X_0 in L^1. Let $Y_t^n = f_n$, the process constant through time. For each n, Y^n is a bounded martingale and $Y^n = (Y^n)^0 = f_n$. Hence, $\|X^0 - Y^n\|_{H^1} = \|X_0 - f_n\|_{L^1}$ and (Y^n) converges to X^0 in H^1. Since $X - X_0$ is in H^1, we may assume now that X is zero at 0.

Let $T_n = \inf\{t\colon\ X_{t-}^* \geq n\}$. Then T_n are predictable stopping times which increase to infinity as n tends to infinity. Since T_n are predictable, $Y^n = X^{T_n-}$ is a local martingale. Moreover, Y^n is bounded by n for each n.

Since T_n increase to infinity, $([X, X]_\infty - [X, X]_{T_n-})^{1/2}$ converges pointwise to zero. Moreover, $([X, X]_\infty - [X, X]_{T_n-})^{1/2} \leq [X, X]_\infty^{1/2}$. But $[X, X]_\infty^{1/2}$ is integrable, since H is in H^1; hence, by dominated convergence

$$\lim_{n\to\infty} E[([X, X]_\infty - [X, X]_{T_n-})^{1/2}] = 0.$$

Then (Y^n) converges to X in H^1, since, by Theorem 3.8 (c),

$$[X-Y^n, X-Y^n]_\infty^{1/2} = ([X, X]_\infty - 2[X, X^{T_n -}]_\infty + [X^{T_n -}, X^{T_n -}]_\infty)^{1/2}$$

$$= ([X, X]_\infty - 2[X, X]_{T_n -} + [X, X]_{T_n -})^{1/2}$$

$$= ([X, X]_\infty - [X, X]_{T_n -})^{1/2}.$$

4 DEFINITION. Let Y be an E-valued uniformly integrable martingale and let $Y_{0-} = 0$. We say Y belongs to BMO^p if there exists a nonnegative constant c such that $E(|Y_\infty - Y_{T-}|^p|F_T) \le c^p$ a.s. for every stopping time T. The infimum over all constants c with this property is denoted by $\|Y\|_{BMO^p}$.

We again note some immediate results.

(a) For $1 \le p_1 \le p_2$, $BMO^{p_2} \subseteq BMO^{p_1}$.

(b) If Y is in BMO^p, then for every stopping time T,

$E(|Y_\infty - Y_{T-}|^p|F_T) \le (\|Y\|_{BMO^p})^p$ a.s.

(c) Let Y be in BMO^2 and let $c = \|Y\|_{BMO^2}$. Taking T = 0 we have $E(|Y_\infty|^2|F_0) \le c^2$. Hence, $E[|Y_\infty|^2] \le c^2$. That is Y_∞ is in L^2 and $BMO^2 \subseteq M_E^2$.

5 THEOREM. Let Y be an E-valued uniformly integrable martingale. Suppose there exists $c \ge 0$ such that for every t,

$E(|Y_\infty - Y_t|^2|F_t) \leq c^2$ a.s. and $|\Delta Y| \leq c$. Then Y is in BMO^2 and $\|Y\|_{BMO^2} \leq \sqrt{2}\, c$.

Proof: We note that, by taking $t = 0$, Y is in M_E^2. Let T be an elementary stopping time taking only a finite number of values $\{t_1,\ldots,t_n\}$. Then for every set A in F_T,

$$\begin{aligned}\int_A |Y_\infty - Y_T|^2 dP &= \int_A \sum_{i=1}^n |Y_\infty - Y_{t_i}|^2 1_{(T=t_i)} dP \\ &= \sum_{i=1}^n \int 1_A 1_{(T=t_i)} |Y_\infty - Y_{t_i}|^2 dP \\ &\leq \sum_{i=1}^n \int 1_A 1_{(T=t_i)} c^2 dP \\ &= \int_A c^2 dP.\end{aligned}$$

Hence,

$$E(|Y_\infty - Y_T|^2|F_T) \leq c^2 \text{ a.s.} \tag{5.1}$$

If T is an arbitrary stopping time, we may choose a sequence of elementary stopping times (T_n) which decrease to T [3, IV.58]. Since Y is square integrable, we have that $|Y_\infty - Y_{T_n}|^2$ converges pointwise and in L^1 to $|Y_\infty - Y_T|^2$, as n tends to infinity. Thus, almost surely

$$\begin{aligned}E(|Y_\infty - Y_T|^2|F_T) &= \lim_n E(|Y_\infty - Y_{T_n}|^2|F_T) \\ &= \lim_n E(E(|Y_\infty - Y_{T_n}|^2|F_{T_n})|F_T)\end{aligned} \tag{5.2}$$

$$\leq \lim_{n} E(c^2 | F_T)$$

$$= c^2.$$

Moreover, by our hypothesis on Y, we have

(5.3) $$|\Delta Y_T| \leq c \text{ a.s.}$$

We next note that

(5.4) $$E(Y_\infty - Y_T | F_T) = E(Y_\infty | F_T) - Y_T = 0 \text{ a.s.}$$

Combining (5.2), (5.3), and (5.4), we have a.s.,

(5.5) $$\begin{aligned} E(|Y_\infty - Y_{T-}|^2 | F_T) &= E(|Y_\infty - Y_T + \Delta Y_T|^2 | F_T) \\ &= E((Y_\infty - Y_T + \Delta Y_T,\ Y_\infty - Y_T + \Delta Y_T) | F_T) \\ &= E(|Y_\infty - Y_T|^2 | F_T) + E(|\Delta Y_T|^2 | F_T) \\ &\leq 2c^2 \\ &= (\sqrt{2}\, c)^2. \end{aligned}$$

Hence, Y is in BMO^2 and $\|Y\|_{BMO^2} \leq \sqrt{2}\, c$.

The following result is completely analogous to the real-valued case with an identical proof [4, VII.86].

6 THEOREM. Let X and Y be E-valued semimartingales and let H and K be two real optional processes. Suppose there exists a nonnegative

constant c such that for every stopping time T,

$$E(\int_{[T,\infty]} K_s^2 d[Y, Y]_s | F_T) \leq c^2 \text{ a.s.}$$

Then, $E[\int^\infty |H_s||K_s||d[X, Y]_s|] \leq c\sqrt{2}\, E[(\int^\infty H_s^2 d[X, X]_s)^{1/2}]$.

The following result is classically known as Fefferman's inequality.

7 COROLLARY. Let X be an E-valued local martingale and let Y be in BMO^2. Then

$$E[\int^\infty |d[X, Y]_s|] \leq \sqrt{2}\, \|Y\|_{BMO^2} \|X\|_{H^1}.$$

Proof: From (5.5), we have for every stopping time T that almost surely

$$\begin{aligned} (7.1) \quad E(|Y_\infty - Y_{T-}|^2 | F_T) &= E(|Y_\infty - Y_T|^2 | F_T) + E(|\Delta Y_T|^2 | F_T) \\ &= E((Y_\infty - Y_T, Y_\infty - Y_T) | F_T) + E(|\Delta Y_T|^2 | F_T) \\ &= E((Y_\infty, Y_\infty) | F_T) - E((Y_T, Y_T) | F_T) \\ &\quad + E(|\Delta Y_T|^2 | F_T). \end{aligned}$$

We next note that Y is in M_E^2, since it is in BMO^2; hence, by Theorem 3.5, the process $(Y_t, Y_t) - [Y, Y]_t$ is a uniformly integrable martingale which is zero at 0. Hence, for every stopping time T

$$(7.2) \quad E((Y_\infty, Y_\infty) - [Y, Y]_\infty | F_T) = (Y_T, Y_T) - [Y, Y]_T \text{ a.s.}$$

Hence,

$$(7.3)\qquad E((Y_\infty, Y_\infty) - (Y_T, Y_T)|F_T) = E([Y, Y]_\infty - [Y, Y]_T|F_T) \text{ a.s.}$$

Finally, we have almost surely that

$$\begin{aligned}(7.4)\qquad E(\int_{[T,\infty]} d[Y, Y]_s|F_T) &= E([Y, Y]_\infty - [Y, Y]_{T-}|F_T) \\ &= E([Y, Y]_\infty - [Y, Y]_T + \Delta[Y, Y]_T|F_T) \\ &= E((Y_\infty, Y_\infty) - (Y_T, Y_T)|F_T) \\ &\quad + E(|\Delta Y_T|^2|F_T) \\ &= E(|Y_\infty - Y_{T-}|^2|F_T) \\ &\le (\|Y\|_{BMO^2})^2.\end{aligned}$$

The result follows by taking $H = 1$ and $K = 1$ in Theorem 6.

8 THEOREM. The dual of H^1 is BMO^2. More precisely, if H^1 and BMO^2 are put in duality by the continuous bilinear form $(X, Y) \to E[[X, Y]_\infty]$, then every continuous linear form J on H^1 arises from a unique element Y in BMO^2 and $\|Y\|_{BMO^2} \le 3\|J\|$.

By the results of this section and the properties of the square bracket, the proof of the analogous result for real-valued processes [4, VII.88] extends immediately to Theorem 8, by appropriately replacing products by inner products.

CHAPTER III
THE OPTIONAL INTEGRAL WITH RESPECT TO A HILBERT PROCESS

In this chapter, we define and characterize the stochastic integral of a real, optional process H with respect to a Hilbert-valued special semimartingale X. This integral will agree with the predictable stochastic integral when H is predictable. The essential problem is to define the integral with respect to a square integrable martingale and then to naturally extend the concept to certain classes of local martingales and processes of finite variation and then to all special semimartingales. The approach in this chapter will parallel that of Déllacherie and Meyer [4] for real-valued X. In Chapter V, we shall see an alternate, perhaps more natural and intuitive, approach. Unless otherwise specified, throughout this chapter H will be real-valued and optional and X will be E-valued.

1. The Optional Integral With Respect to a Square Integrable Martingale

We begin by letting X be a square integrable martingale. We shall define an extension of the predictable stochastic integral to optional processes and then study its properties.

1 DEFINITION. Let $L_0^2(X)$ be the space of real, optional processes H such that

$$\|H\|_X = (E[\int^\infty H_s^2 d[X, X]_s])^{1/2} < \infty.$$

Since $\langle X, X\rangle = [X, X]^p$, if H is predictable we then have $E[\int^\infty H_s^2 d[X, X]_s] = E[\int^\infty H_s^2 d\langle X, X\rangle_s]$. Thus, $L_p^2(X)$ is the subspace of $L_0^2(X)$ consisting of predictable H.

2 THEOREM. Let H be in $L_0^2(X)$. There exists a unique E-valued square integrable martingale $H_c^{\bullet}X$, called the optional (or compensated) stochastic integral of H with respect to X, such that for every Y_∞ in $L^2(\Omega, F, P, E)$

$$E[((H_c^{\bullet}X)_\infty, Y_\infty)] = E[\int^\infty H_s d[X, Y]_s],$$

where Y is a cadlag version of the square integrable martingale $E(Y_\infty|F_t)$.

Proof: We define a mapping from the Hilbert space $L^2(\Omega, F, P, E)$, which may be identified with its dual, into the real line as follows: for every Y_∞ in L^2, let

(2.1) $$J(Y_\infty) = E[\int^\infty H_s d[X, Y]_s].$$

By the Kunita-Watanabe inequality (II.4.1) followed by Holder's inequality, we have

(2.2) $$\begin{aligned} E[\int^\infty |H_s||d[X, Y]_s|] &\le E[(\int^\infty H_s^2 d[X, X]_s)^{1/2}(\int^\infty d[Y, Y]_s)^{1/2}] \\ &\le (E[\int^\infty H_s^2 d[X, X]_s])^{1/2}(E[[Y, Y]_\infty])^{1/2} \\ &= \|H\|_X\|Y\|_{M^2} \end{aligned}$$

$$= \|H\|_X \|Y_\infty\|_{L^2}$$

$$< \infty.$$

Thus, $\int^\infty |H_s||d[X, Y]_s|$ is integrable and hence finite almost surely. Outside of an evanescent set, we then have that process $\int^t H_s d[X, Y]_s$ is defined and is integrable for every $t \leq \infty$. In particular, $J(Y_\infty)$ is well-defined. By the linearity of the square bracket and the two integrals, it follows that J is linear. Moreover J is continuous, since $\|H\|_X < \infty$ and

$$|J(Y_\infty)| \leq E[\int^\infty |H_s||d[X, Y]_s|] \leq \|H\|_X \|Y_\infty\|_{L^2}.$$

Hence, there exists a unique W_∞ in $L^2(\Omega, F, P, E)$ such that for every Y_∞ in $L^2(\Omega, F, P, E)$

$$E[(W_\infty, Y_\infty)] = J(Y_\infty) = E[\int^\infty H_s d[X, Y]_s]. \tag{2.3}$$

We then let $H \underset{c}{\cdot} X$ be a cadlag version of the square integrable martingale $E(W_\infty | F_t)$. Up to indistinguishability, $H \underset{c}{\cdot} X$ is unique, for if Z were another such square integrable martingale, then Z_∞ is in L^2. Hence, if $E[(Z_\infty, Y_\infty)] = E[\int^\infty H_s d[X, Y]_s]$ for every Y_∞ in L^2, then by (2.3), $Z_\infty = W_\infty$ a.s., and then Z and $H \underset{c}{\cdot} X$ are indistinguishable.

We recall that the space of E-valued square integrable martingales M_E^2 is isometrically isomorphic to $L^2(\Omega, F, P, E)$; hence, the theorem may be worded to say $E[((H \underset{c}{\cdot} X)_\infty, Y_\infty)] = E[\int^\infty H_s d[X, Y]_s]$ for every Y in M_E^2, where Y_∞ is the limit at infinity of Y.

3 COROLLARY. Let H be in $L_0^2(X)$. For every f in E, $(f, H\underset{c}{\cdot}X) = H\underset{c}{\cdot}(f, X)$.

Proof: We note that Theorem 2 also characterizes $H\underset{c}{\cdot}X$ if X is real, by taking E to be the real line. By Theorem II.4.9,

$$\int^\infty H_s^2 d[(f, X), (f, X)]_s \le |f|^2 \int^\infty H_s^2 d[X, X]_s;$$

hence, H is in $L_0^2((f, X))$ and $H\underset{c}{\cdot}(f, X)$ exists. Then using Theorem II.3.9, we have for every Y in M^2,

$$\begin{aligned} E[\int^\infty H_s d[(f, X), Y]_s] &= E[\int^\infty H_s d[X, fY]_s \\ &= E[((H\underset{c}{\cdot}X)_\infty, (fY)_\infty)] \\ &= E[((H\underset{c}{\cdot}X)_\infty, f)Y_\infty]. \end{aligned}$$

Hence, $H\underset{c}{\cdot}(f, X) = (H\underset{c}{\cdot}X, f) = (f, H\underset{c}{\cdot}X)$.

4 THEOREM. Let H be in $L_0^2(X)$ and let $W = H\underset{c}{\cdot}X$.

(a) If H is predictable, then $W = H\cdot X$.

(b) $E[[W, W]_\infty] \le E[\int^\infty H_s^2 d[X, X]_s]$.

(c) For every Y in M_E^2, the process $A = [W, Y] - H\cdot[X, Y]$ is a uniformly integrable martingale, which is zero at 0, and it is moreover a process of integrable variation. If H is predictable, then A is identically zero.

(d) $W_0 = H_0X_0$ a.s.; $\Delta W = H\Delta W - {}^p(H\Delta X)$ on $]0, \infty[$.

(e) $W^T = H\underset{c}{\cdot}X^T = H1_{[0,T]}\underset{c}{\cdot}X$ for every stopping time T.

Proof:

(a) Let Y_∞ be in L^2. We saw in Theorem II.4.3 that $\langle X, X\rangle = [X, X]^p$. Since X and Y are both square integrable, by polarization $\langle X, Y\rangle$ exists and $\langle X, Y\rangle = [X, Y]^p$. Then if H is predictable, it follows by Theorem II.1.4, that

$$\begin{aligned} E[(W_\infty, Y_\infty)] &= E[\int^\infty H_s d[X, Y]_s] \\ &= E[\int^\infty H_s d\langle X, Y\rangle_s] \\ &= E[((H\cdot X)_\infty, Y_\infty)]. \end{aligned}$$

Hence, $W = H\cdot X$ by uniqueness.

(b) Using Corollary II.3.6, we have

$$\begin{aligned} E[[W, W]_\infty] &= E[(W_\infty, W_\infty)] \\ &= E[\int^\infty H_s d[X, W]_s] \\ &\leq E[\int^\infty |H_s||d[X, W]_s|]. \end{aligned}$$

Then, using inequality (2.2), we have

$$0 \leq E[[W, W]_\infty] \leq \|H\|_X (E[[W, W]_\infty])^{1/2}.$$

If $E[[W, W]_\infty] = 0$, then (b) trivially holds. Otherwise, by dividing, we obtain

$$(E[[W, W]_\infty])^{1/2} \leq \|H\|_X.$$

Then squaring both sides yields (b).

(c) We may write A as $\int^t dA_s = \int^t d[W, Y]_s + \int^t H_s d[X, Y]_s$. Applying Theorem II.4.1 and Holder's inequality we obtain, as in (2.2), that

$$E[\int^\infty |dA_s|] \leq E[\int^\infty |d[W, Y]_s|] + E[\int^\infty |H_s||d[X, Y]_s|]$$

$$\leq \|W_\infty\|_{L^2}\|Y_\infty\|_{L^2} + \|H\|_X\|Y_\infty\|_{L^2} < \infty.$$

Thus, A is of integrable variation. Moreover, the above inequality shows that [W, Y] and H•[X, Y] separately are of integrable variation. Then, since A_∞ is integrable, it will suffice to show that for every stopping time T, $E[A_T] = c$, for some constant c independent of T.

For every T, we have, by Theorem II.3.8 and by a property of the Stieltjes integral, that

$$[W, Y^T]_\infty = [W, Y]^T_\infty = [W, Y]_T$$

and

$$\int^T H_s d[X, Y]_s = \int^\infty H_s d[X, Y]^T_s = \int^\infty H_s d[X, Y^T]_s.$$

Since Y^T is still in M^2, we have, by Corollary II.3.6,

$$E[[W, Y]_T] = E[[W, Y^T]_\infty]$$

$$= E[(W_\infty, Y^T_\infty)]$$

$$= E[\int^\infty H_s d[X, Y^T]_s]$$

$$= E[\int^T H_s d[X, Y]_s].$$

Hence, for every stopping time T

$$E[A_T] = E[[W, Y]_T] - E[\int^T H_s d[X, Y]_s] = 0.$$

Thus, A is a uniformly integrable martingale. Also

$$\begin{aligned} A_0 &= [W, Y]_0 - H_0[X, Y]_0 \\ &= (W_0, Y_0) - H_0(X_0, Y_0) \\ &= (W_0, Y_0) - (H_0X_0, Y_0). \end{aligned}$$

Then to show $A_0 = 0$, it suffices to show $W_0 = H_0X_0$, which is assertion (d). If H is predictable, then A = 0, by (a) and Theorem II.4.6.

(d) The result is known for real-valued X [4, VIII.33]. If f is in E, then almost surely,

$$\begin{aligned} (f, W_0) &= (f, H_c^{\bullet}X)_0 = (H_c^{\bullet}(f, X))_0 \\ &= H_0(f, X)_0 = H_0(f, X_0) = (f, H_0X_0). \end{aligned}$$

Letting f range through a countable norming set of the separable Hilbert space E, we have $W_0 = H_0X_0$ a.s.

The second assertion in (d) may also be proven weakly. Beforehand though, we verify that ${}^p(H\Delta X)$ exists. By Brooks and Dinculeanu [2], it suffices to show ${}^p|H\Delta X| < \infty$, outside an evanescent set. For every predictable time T,

$$
\begin{aligned}
E[{}^p|H\Delta X|_T 1_{(T<\infty)}] &= E[|H_T\Delta X_T| 1_{(T<\infty)}] \\
&= E[|H_T||\Delta X_T|] \\
&= E[(\int_{[T]} H_s^2 d[X, X,]_s)^{1/2}] \\
&\le (E[\int_{[T]} H_s^2 d[X, X]_s])^{1/2} \\
&\le \|H\|_X .
\end{aligned}
$$

Hence, by the cross-section theorem [3, IV.85] we have, outside of an evanescent set, that ${}^p|H\Delta X| \le |H|_X < \infty$ and ${}^p(H\Delta X)$ exists. Then for every f in E, outside of an evanescent set, we have on $]0, \infty[$

$$
\begin{aligned}
(f, \Delta W) &= \Delta(f, W) \\
&= \Delta(H\underset{c}{\cdot}(f, X)) \\
&= H\Delta(f, X) - {}^p(H\Delta(f, X)) \\
&= (f, H\Delta X) - {}^p(f, H\Delta X) \\
&= (f, H\Delta X) - (f, {}^p(H\Delta X)) \\
&= (f, H\Delta X - {}^p(H\Delta X)).
\end{aligned}
$$

Letting f range through a countable norming set, we have that ΔW and $H\Delta X - {}^p(H\Delta X)$ are indistinguishable on $]0, \infty[$.

(e) For every stopping time T,

$$E[\int^{\infty} H_s^2 d[X^T, X^T]_s] = E[\int^{\infty} H_s^2 d[X, X]_s^T]$$

$$\leq E[\int^{\infty} H_s^2 d[X, X]_s]$$

and

$$E[\int^{\infty} H_s^2 1_{[0,T]} d[X, X]_s] \leq E[\int^{\infty} H_s^2 d[X, X]_s]$$

so that $H \underset{c}{\cdot} X^T$ and $H1_{[0,T]} \underset{c}{\cdot} X$ exist.

If $W = H \underset{c}{\cdot} X$, then $W_{\infty}^T = W_T$ and for every Y in M^2

$$E[(W_T, Y_{\infty})] = E[(E(W_{\infty}|F_T), Y_{\infty})]$$

$$= E[E((W_{\infty}, Y_{\infty})|F_T)]$$

$$= E[(W_{\infty}, E(Y_{\infty}|F_T))]$$

$$= E[(W_{\infty}, Y_T)]$$

$$= E[\int^{\infty} H_s d[X, Y^T]_s]$$

$$= E[\int^{\infty} H_s d[X^T, Y]_s]$$

$$= E[((H \underset{c}{\cdot} X^T)_{\infty}, Y_{\infty})].$$

Hence, $W^T = H \underset{c}{\cdot} X^T$. We also have that

$$E[(W_T, Y_{\infty})] = E[\int^{\infty} H_s d[X, Y^T]_s]$$

$$= E[\int^{\infty} H_s d[X, Y]_s^T]$$

$$= E[\int^{\infty} H_s 1_{[0,T]} d[X, Y]_s]$$

$$= E[((H1_{[0,T]} \underset{c}{\bullet} X)_\infty, Y_\infty)].$$

Hence, $W^T = H1_{[0,T]} \underset{c}{\bullet} X$.

5 THEOREM.

(a) If H is in $L_0^2(X)$ and $L_0^2(Z)$, then H is in $L_0^2(X+Z)$ and $H \underset{c}{\bullet} (X+Z) = H \underset{c}{\bullet} X + H \underset{c}{\bullet} Z$.

(b) If H and K are in $L_0^2(X)$, then H+K is in $L_0^2(X)$ and $(H+K) \underset{c}{\bullet} X = H \underset{c}{\bullet} X + K \underset{c}{\bullet} X$.

Proof:

(a) Using the bilinearity of the square bracket, we obtain

$$E[\int^{\infty} H_s^2 d[X+Z, X+Z]_s] = E[\int^{\infty} H_s^2 d[X, X]_s]$$

$$+ 2E[\int^{\infty} H_s^2 d[X, Z]_s] + E[\int^{\infty} H_s^2 d[Z, Z]_s]$$

$$\leq \|H\|_X^2 + 2E[\int^{\infty} H_s^2 |d[X, Z]_s|] + \|H\|_Z^2.$$

Then by Theorem II.4.1 followed by Holder's inequality,

$$E[\int^{\infty} |H_s||H_s||d[X, Z]_s|] \leq \|H\|_X \|H\|_Z.$$

Hence,

$$\|H\|_{X+Z} \leq (\|H\|_X^2 + 2\|H\|_X \|H\|_Z + \|H\|_Z^2)^{1/2}$$

$$= \|H\|_X + \|H\|_Z.$$

Let $W = H{\underset{c}{\cdot}}X$, $U = H{\underset{c}{\cdot}}Z$, and $V = H{\underset{c}{\cdot}}(X+Z)$. Then for every Y in M^2,

$$\begin{aligned} E[(V_\infty, Y_\infty)] &= E[\int^\infty H_s d[X+Z, Y]_s] \\ &= E[\int^\infty H_s d[X, Y]_s] + E[\int^\infty H_s d[Z, Y]_s] \\ &= E[(W_\infty, Y_\infty)] + E[(U_\infty, Y_\infty)] \\ &= E[(W_\infty + U_\infty, Y_\infty)]. \end{aligned}$$

Hence, $V = W + U$.

(b) By Theorem II.3.3 and Corollary II.3.6, $[X, X]$ is an integrable increasing process. Hence, there exists a unique P-measure m such that for every bounded, measurable process Y, $\int Y dm = E[\int^\infty Y_s d[X, X]_s]$ [4, VI.65]. By truncation and monotone convergence, we have that for every measurable process Y,

$$\int Y^2 dm = E[\int^\infty Y_s^2 d[X, X]_s]. \tag{5.1}$$

Hence, $L_0^2(X)$ is a subspace of $L^2(R^+ \times \Omega, B(R^+) \times F, m, R)$ and, by Minkowski's inequality,

$$\|H + K\|_X \leq \|H\|_X + \|K\|_X.$$

Thus, $H + K$ is in $L_0^2(X)$.

We now let $W = H{\underset{c}{\cdot}}X$, $U = K{\underset{c}{\cdot}}X$, and $V = (H+K){\underset{c}{\cdot}}X$. Then for every Y in M^2,

$$\begin{aligned} E[(V_\infty, Y_\infty)] &= E[\int^\infty (H_s+K_s)d[X, Y]_s] \\ &= E[\int^\infty H_s d[X, Y]_s] + E[\int^\infty K_s d[X, Y]_s] \\ &= E[(W_\infty+U_\infty, Y_\infty)]. \end{aligned}$$

Hence, $V = W + U$.

We now state two immediate convergence theorems for the optional stochastic integral.

6 THEOREM.

(a) Let (H^K) be a sequence of processes in $L_0^2(X)$ which are uniformly bounded in absolute value by a process L in $L_0^2(X)$. If H^K converges pointwise to H, then H is in $L_0^2(X)$ and $H^K \underset{c}{\cdot} X$ converges in M_E^2 to $H \underset{c}{\cdot} X$.

(b) Let H be a bounded, optional process. If (X^n) is a sequence of square integrable martingales converging to X in M_E^2, then $H \underset{c}{\cdot} X^n$ converges to $H \underset{c}{\cdot} X$ in M_E^2.

Proof:

(a) We return to the notations of the proof of Theorem 5 (b). Since H is the limit of optional processes, H is optional. By the dominated convergence theorem in L^2 with respect to the measure m, H is in $L^2(m)$ and H^K converges to H in $L^2(m)$. By (5.1), H^K converges to H in $L_0^2(X)$.

We now let $W^K = H^K \underset{c}{\cdot} X - H \underset{c}{\cdot} X = (H^K - H) \underset{c}{\cdot} X$. By Theorem 4 (b), we have

$$\|W^K\|_{M_E^2} = (E[[W^K, W^K]_\infty])^{1/2}$$

$$\leq \|H^K - H\|_X.$$

Hence, $H^K \underset{c}{\cdot} X$ converges to $H \underset{c}{\cdot} X$ in M_E^2.

(b) Since H is bounded, H is in $L_0^2(X)$ and $L_0^2(X^n)$ for every n. We choose a finite constant c such that $|H| \leq c$. Let $W^n = H \underset{c}{\cdot} X^n - H \underset{c}{\cdot} X = H \underset{c}{\cdot} (X^n - X)$. Then

$$\|W^n\|_{M_E^2} = (E[[W^n, W^n]_\infty])^{1/2}$$

$$\leq (E[\int^\infty H_s^2 d[X^n - X, X^n - X]_s])^{1/2}$$

$$\leq c(E[[X^n - X, X^n - X]_\infty])^{1/2}$$

$$= c\|X^n - X\|_{M_E^2}.$$

Hence, $H \underset{c}{\cdot} X^n$ converges to $H \underset{c}{\cdot} X$ in M_E^2.

Another application of Corollary 3 is the following "associativity" theorem which is again known for real X.

7 THEOREM. Let H be in $L_0^2(X)$. If K is real-valued and is in both $L_p^2(X)$ and $L_p^2(H \underset{c}{\cdot} X)$ and $E[\int^\infty H_s^2 K_s^2 d[X, X]_s] < \infty$, then $H \underset{c}{\cdot} (K \cdot X) = HK \underset{c}{\cdot} X = K \cdot (H \underset{c}{\cdot} X)$.

Proof: We first verify that all three integrals are defined. $K\cdot X$ is defined, since K is in $L_p^2(X)$, and applying II.4.7, we have

$$E[\int^\infty H_s^2 d[K\cdot X, K\cdot X]_s] = E[\int^\infty H_s^2 K_s^2 d[X, X]_s] < \infty.$$

Hence, H is in $L_0^2(K\cdot X)$ and HK is in $L_0^2(X)$ and the first two integrals are defined. The third integral is defined by hypothesis.

Let f be in E. Then, outside of an evanescent set, we have

$$\begin{aligned}(f, H\underset{c}{\cdot}(K\cdot X)) &= H\underset{c}{\cdot}(f, K\cdot X) \\ &= H\underset{c}{\cdot}(K\cdot(f, X)) \\ &= HK\underset{c}{\cdot}(f, X) \\ &= (f, HK\underset{c}{\cdot}X)\end{aligned}$$

and

$$\begin{aligned}(f, HK\underset{c}{\cdot}X) &= HK\underset{c}{\cdot}(f, X) \\ &= K\cdot(H\underset{c}{\cdot}(f, X)) \\ &= K\cdot(f, H\underset{c}{\cdot}X) \\ &= (f, K\cdot(H\underset{c}{\cdot}X)).\end{aligned}$$

Letting f range through a countable norming set in E, we obtain, outside of one evanescent set, that

$$H\underset{c}{\cdot}(K\cdot X) = HK\underset{c}{\cdot}X = K\cdot(H\underset{c}{\cdot}X).$$

8 THEOREM. Let E_0 be a closed subspace of E. If X takes values in E_0, then $H \underset{c}{\cdot} X$ also takes values in E_0 for every H in $L_0^2(X)$.

Proof: We define a continuous, linear mapping J_1 from $L^2(\Omega, F, P, E_0)$ into the real line in the manner of the proof of Theorem 2. We let for every Y_∞ in $L^2(\Omega, F, P, E_0)$

$$J_1(Y_\infty) = E[\int^\infty H_s d[X, Y]_s,$$

where Y is a cadlag version of the square integrable martingale $E(Y_\infty | F_t)$.

Then there exists a unique element W_∞^1 in $L^2(\Omega, F, P, E_0)$ such that for every Y_∞ in $L^2(\Omega, F, P, E_0)$

$$E[(W_\infty^1, Y_\infty)] = E[\int^\infty H_s d[X, Y]_s].$$

The square integrable martingale $W_t^1 = E(W_\infty^1 | F_t)$ (a cadlag version) then takes values in E_0. We assert that $W^1 = H \underset{c}{\cdot} X$. Let W denote $H \underset{c}{\cdot} X$,

Let Y_∞ be in $L^2(\Omega, F, P, E)$. We then may write $Y_\infty = Y_\infty^1 + Y_\infty^2$, where Y_∞^1 is in $L^2(\Omega, F, P, E_0)$ and Y_∞^2 is in $L^2(\Omega, F, P, E_0^\perp)$. By Theorem I.3.10, $[X, Y^2] = 0$ and hence $[X, Y] = [X, Y^1]$. Thus,

$$\begin{aligned} E[(W_\infty, Y_\infty)] &= E[\int^\infty H_s d[X, Y]_s] \\ &= E[\int^\infty H_s d[X, Y^1]_s] \\ &= E[(W_\infty^1, Y_\infty^1)] \\ &= E[(W_\infty^1, Y_\infty)]. \end{aligned}$$

The last equality follows from the fact that $(W^1_\infty, Y^2_\infty) = 0$. Hence, $W = W^1$ by the uniqueness of Theorem 2.

Theorem 8 merits some more discussion. If X_∞ is an integrable random vector, then it is the almost sure limit of a sequence of simple random vectors, each of which takes a finite number of values. If we let E_1 denote the union of the ranges of such a sequence and let E_0 be the closed linear span of E_1, then X_∞ takes its values almost surely in E_0, which is a closed subspace of our original space which we have been denoting by E. Moreover, E_0 is separable.

Now let X_t denote a cadlag version of the martingale $E(X_\infty|F_t)$. If we let (X^n_∞) be a sequence of simple random vectors converging to X_∞ in $L^1(\Omega, F, P, E_0)$, then $E(X^n_\infty|F_t)$ will also converge to $E(X_\infty|F_t)$ in $L^1(\Omega, F, P, E_0)$ for every t. Since X^n_∞ are simple, it is clear that $E(X^n_\infty|F_t)$ takes values in E_0 for all n and t.

We fix a rational t. Through an appropriate subsequence, $E(X^n_\infty|F_t)$ converges a.s. to $E(X_\infty|F_t)$. Hence, almost surely

$$X_t = E(X_\infty|F_t) = \lim_n E(X^n_\infty|F_t).$$

Hence, X_t also takes values in E_0, outside of a null set. By right continuity and taking limits along the rationals, we have that X takes values in E_0 outside of an evanescent set.

Therefore, in constructing the optional stochastic integral, we need not assume that E is separable. This hypothesis was not used in the proof of Theorem 2, and the above discussion along with Theorem 8 shows that we may reduce the general case to one which includes separability as a hypothesis.

9 THEOREM. Let H be in $L_0^2(X)$ and $W = H \underset{c}{\cdot} X$. If X is quasi-left continuous, then we have the following:

(a) $\Delta W = H\Delta X$.

(b) $[H \underset{c}{\cdot} X, Y] = H \cdot [X, Y]$, for every Y in M_E^2.

(c) Theorem 7 remains valid for optional K.

Proof: With the convention that $X_{0-} = X_0$, we have that for every predictable time T, $\Delta X_T = 0$ a.s. The result is known for real X [4, VIII.35].

(a) Let f be in E. Outside of an evanescent set, since (f, X) is still quasi-left continuous, we have

$$(f, \Delta W) = \Delta(f, W) = \Delta H \underset{c}{\cdot} (f, X) = H\Delta(f, X) = (f, H\Delta X).$$

Hence, ΔW and $H\Delta W$ are indistinguishable in the separable space E.

(b) By Theorem 4 (c), we know that for every Y in M^2 $A = [W, Y] - H \cdot [X, Y]$ is a uniformly integrable martingale and a process of integrable variation. Then for every stopping time T, we have, by (a),

$$\begin{aligned} E(A_\infty - A_{T-} | F_T) &= \Delta A_T \\ &= (\Delta W_T, \Delta Y_T) - H_T(\Delta X_T, \Delta Y_T) \\ &= (\Delta W_T - H_T \Delta X_T, \Delta Y_T) \\ &= 0 \text{ a.s.} \end{aligned}$$

Hence, A and 0 have the same optional dual projection [4, VI.75]. Since A is already optional, A = 0.

(c) In the hypothesis of Theorem 7, we make the change that K is in $L_0^2(X)$ and $L_0^2(H\underset{c}{\cdot}X)$. Applying (b) twice, we have

$$[K\underset{c}{\cdot}X,\ K\underset{c}{\cdot}X] = K^2\cdot[X,\ X].$$

As in the proof of 7, we then have that $H\underset{c}{\cdot}(K\underset{c}{\cdot}X)$, $HK\underset{c}{\cdot}X$, and $K\underset{c}{\cdot}(H\underset{c}{\cdot}X)$ are all defined. The remainder of the proof of 7 remains valid since the result for real X is known.

2. Relation to the Stieltjes Integral

In Theorem II.2.2, we saw for the predictable stochastic integral that for locally bounded H in $L_p^2(X)$, where X is also a process of finite variation, H·X is the usual Stieltjes integral. For the optional integral, we do not obtain an analogous result. In the discussion following Theorem 4.8, we shall see further why this is the case.

1 LEMMA. Let H be in $L_0^2(X)$. Then $H\underset{c}{\cdot}X = H\underset{c}{\cdot}(X-X_0) + H_0X_0$.

Proof: We first note that

$$E[\int^\infty H_s^2 d[X-X_0,\ X-X_0]_s] = E[\int^\infty H_s^2 d([X,\ X]_s - [X,\ X]_0)]$$

$$\leq E[\int^\infty H_s^2 d[X,\ X]_s].$$

Hence, H is in $L_0^2(X-X_0)$. Thus, by Theorem 1.5,

$$H_c^{\bullet}X - H_c^{\bullet}(X-X_0) = H_c^{\bullet}X_0 = H_c^{\bullet}X^0 = (H_c^{\bullet}X)^0$$

$$= (H_c^{\bullet}X)_0 = H_0X_0.$$

The preceding lemma allows us, when convenient, to work with $X - X_0$ and thereby assume from the start that X is zero at 0.

2 LEMMA. Let X be an E-valued local martingale which is zero at 0 and let H be a real optional process. If X is also a process of finite variation and if the process $\int^t |H_s||dX_s|$ is locally integrable then, letting $A_t = \int^t H_s dX_s$ and $N_t = A_t - A_t^p$, we have for every E-valued local martingale Y that the process $[N, Y]_t - \int^t H_s d[X, Y]_s$ is a local martingale which is zero at 0.

Proof: We first verify that the process $\int^t H_s d[X, Y]_s$ is defined, outside of an evanescent set.

Since $\int^t |H_s||dX_s|$ is locally integrable and zero at 0, there exists a sequence of stopping times (T_n) such that $\int^{T_n} |H_s||dX_s|$ is integrable and hence finite almost surely. Since [Y, Y] is increasing, we may also assume that [Y, Y] is bounded by n on $[0, T_n[$.

Since X is of finite variation, $[X, X]_t$ reduces to the sum of the squares of the jumps of X; hence, for every measurable process K,

$$(2.1) \qquad \int^t |K_s| d[X, X]_s = \sum_{s \le t} |K_s||\Delta X_s|^2.$$

By the Kunita-Watanabe inequality, we have

$$\int_{[0,T_n[}|H_s||d[X,Y]_s| \le \Big(\int_{[0,T_n[}|H_s|^2 d[X,X]_s\Big)^{1/2}\Big(\int_{[0,T_n[} d[Y,Y]_s\Big)^{1/2}$$

$$(2.2) \qquad = \Big(\sum_{s<T_n}|H_s|^2|\Delta X_s|^2\Big)^{1/2}\Big(\int_{[0,T_n[} d[Y,Y]_s\Big)^{1/2}$$

$$\le n^{1/2}\sum_{s<T_n}|H_s||\Delta X_s|$$

$$\le n\int^{T_n}|H_s||dX_s|$$

$$< \infty \quad \text{a.s.}$$

Since T_n increases to infinity, the process $\int^t|H_s||d[X,Y]_s|$ is finite outside of an evanescent set and $\int^t H_s d[X,Y]_s$ is defined.

Now since X is of finite variation, [X, Y] is a sum of jumps, by Theorem II.3.4; thus,

$$\int^t H_s d[X,Y]_s = \sum_{s\le t} H_s(\Delta X_s, \Delta Y_s)$$

$$= \sum_{s\le t}(H_s\Delta X_s, \Delta Y_s)$$

$$= \sum_{s\le t}(\Delta A_s, \Delta Y_s)$$

$$= [A, Y]_t.$$

The last equality holds since $A_t = \int^t H_s dX_s$ is also a process of finite variation. We then have that

$$[N, Y]_t - \int^t H_s d[X, Y]_s = -[A^p, Y]_t$$

is a local martingale by Theorem II.3.4, since A^p is a predictable process of finite variation and Y is a local martingale.

Finally, since $A_0^p = A_0 = H_0X_0 = 0$, we have that $-[A^p, Y]_0 = -(A_0^p, Y_0) = 0$, and the above local martingale is zero at 0. (We recall in general that since A is locally integrable, A^p is the unique predictable process of finite variation such that $A - A^p$ is a local martingale which is zero at 0.)

3 THEOREM. Let H be in $L_0^2(X)$ and assume that X is also a process of finite variation. If the process $\int^t |H_s||dX_s|$ is finite outside of an evanescent set, then this process is in fact locally integrable and $H \underset{c}{\cdot} X = A - A^p + H_0X_0$, where $A_t = \int^t H_s dX_s$.

Proof: We first recall that to show $\int^t |H_s||dX_s|$ is locally integrable, we must show that the process $\int^t |H_s||dX_s| - |H_0||X_0|$ is locally integrable. However,

$$\begin{aligned}\int^t |H_s||dX_s| - |H_0||X_0| &= \int^t |H_s|(|dX_s| - |dX_0|) \\ &\leq \int^t |H_s||d(X_s - X_0)| \\ &\leq \int^t |H_s|(|dX_s| + |dX_0|) \\ &\leq \int^t |H_s||dX_s| + |H_0||X_0| \\ &= \int^t |H_s||dX_s| + \int_{[0]} |H_s||dX_s|.\end{aligned}$$

Hence, it suffices to show that the process $\int^t |H_s||d(X_s - X_0)|$ is locally integrable, since this process dominates $\int^t |H_s||dX_s| - |H_0||X_0|$ and is also finite by the last equality. By working with $X - X_0$, we may assume that $X_0 = 0$. (We have noted in Lemma 1 that H is also in $L_0^2(X - X_0)$.)

Then to show $\int^t |H_s||dX_s|$ is locally integrable, it suffices to show that the process has bounded jumps, since it is increasing.

We have for every stopping time T,

$$\begin{aligned} E[\Delta\int^T |H_s||dX_s|] &= E[|H_T||\Delta X_T|] \\ &= E[(|H_T^2|\Delta[X, X]_T)^{1/2}] \\ &\le (E[|H_T^2|\Delta[X, X]_T)^{1/2} \\ &= (E[\int_{[T]} H_s^2 d[X, X]_s)^{1/2} \\ &\le \|H\|_X. \end{aligned}$$

By the cross-section theorem, the jumps of $\int^T |H_s||dX_s|$ are bounded, outside of an evanescent set, by $\|H\|_X$.

We now let $A_t = \int^t H_s dX_s$, and $N_t = A_t - A_t^p$. We assert that N is a square integrable martingale and that $N = H \underset{c}{\cdot} X$.

Using Corollary II.4.2 and Theorem II.4.4 we have

$$E[[N, N]_\infty] \le 2E[[A, A]_\infty] + 2E[[-A^p, -A^p]_\infty]$$

$$= 2E[[A, A]_\infty] + 2E[[A^p, A^p]_\infty]$$

$$\leq 4E[[A, A]_\infty]$$

$$= 4E[\sum_{s<\infty} (\Delta A_s, \Delta A_s)]$$

$$= 4E[\sum_{s<\infty} (H_s\Delta X_s, H_s\Delta X_s)]$$

$$= 4E[\sum_{s<\infty} H_s^2\Delta[X, X]_s]$$

$$\leq 4E[\int^\infty H_s^2 d[X, X]_s]$$

$$< \infty.$$

Hence, by Theorem II.3.7, N is a square integrable martingale.

Let Y be in M_E^2. Then by Lemma 2, $[N, Y]_t - \int^t H_s d[X, Y]_s$ is a local martingale which is zero at 0. But by the Kunita-Watanabe inequality and Holder's inequality, we have

$$E[\int^\infty |d[N, Y]_s|] \leq \|N_\infty\|_{L^2}\|Y_\infty\|_{L^2} < \infty$$

and

$$E[\int^\infty |H_s||d[X, Y]_s|] \leq \|H\|_X\|Y_\infty\|_{L^2} < \infty.$$

Hence, $[N, Y]_t - \int^t H_s d[X, Y]_s$ is of integrable variation and is thus a local martingale of class D, which then makes it a true uniformly integrable martingale. Since it is zero at 0, we have

$$E[(N_\infty, Y_\infty)] = E[[N, Y]_\infty]$$

$$= E[\int^\infty H_s d[X, Y]_s]$$

$$= E[((H\underset{c}{\cdot}X)_\infty,\ Y_\infty)].$$

Hence, $N = H\underset{c}{\cdot}X$.

We were working with $X - X_0$, so we have shown that $H\underset{c}{\cdot}(X - X_0) = \int^t H_s d(X_s - X_0) - (\int^{\cdot} H_s d(X_s - X_0))^p$. However,

$$\int^t H_s d(X_s - X_0) - (\int^{\cdot} H_s d(X_s - X_0))^p$$

$$= \int^t H_s dX_s - (\int^{\cdot} H_s dX_s)^p - H_0X_0 + (H_0X_0)^p$$

$$= \int^t H_s dX_s - (\int^{\cdot} H_s dX_s)^p.$$

Hence,

$$H\underset{c}{\cdot}X = H\underset{c}{\cdot}(X - X_0) + H_0X_0$$

$$= \int^t H_s dX_s - (\int^{\cdot} H_s dX_s)^p + H_0X_0.$$

3. The Optional Integral With Respect to a Special Semimartingale

We shall now define and characterize the stochastic integral of a real optional process H with respect to an E-valued special semimartingale X. The class of optional processes for which we define this integral will include locally bounded processes. We shall of course still maintain agreement with the predictable integral when H is locally bounded and predictable.

1 LEMMA. Let X be in M_E^2 and let H be in $L_0^2(X)$. Then $\|H\underset{c}{\cdot}X\|_{H^1} \le 3\sqrt{2}\ E[(\int^\infty H_s^2 d[X,\ X]_s)^{1/2}]$.

Proof: We note in particular that for H bounded and optional, H is in $L^2_0(X)$. In any case, $H_c \cdot X$ is in $M^2_E = H^2$, which is contained in H^1.

Let J be a continuous linear form on H^1. By Theorem II.5.8, there exists a unique Y in BMO^2 such that $J(Z) = E[[Z, Y]_\infty]$, for every Z in H^1. Moreover, $\|Y\|_{BMO^2} \leq 3\|J\|$.

By (7.4) in the proof of Corollary II.5.7, we have that for every stopping time T

$$E[\int_{[T,\infty]} d[Y, Y]_s | F_T) \leq (\|Y\|_{BMO^2})^2 \quad \text{a.s.}$$

Then applying Theorem II.5.6, we obtain

$$E[\int^\infty |H_s||d[X, Y]_s|] \leq \sqrt{2}\, \|Y\|_{BMO^2} E[(\int^\infty H_s^2 d[X, X]_s)^{1/2}].$$

Hence,

$$(1.1) \qquad \begin{aligned} \|H_c \cdot X\|_{H^1} &= \sup_{\|J\| \leq 1} |J(H_c \cdot X)| \\ &\leq \sup_{\substack{Y \text{ in } BMO^2 \\ \|Y\| \leq 3}} |E[[H_c \cdot X, Y]_\infty]| \\ &= \sup_{\substack{Y \text{ in } BMO^2 \\ \|Y\| \leq 3}} |E[\int^\infty H_s d[X, Y]_s]| \\ &\leq \sup_{\substack{Y \text{ in } BMO^2 \\ \|Y\| \leq 3}} E[\int^\infty |H_s||d[X, Y]_s|] \\ &\leq 3\sqrt{2}\, E[(\int^\infty H_s^2 d[X, X]_s)^{1/2}]. \end{aligned}$$

2 DEFINITION. Let X be a special semimartingale with canonical decomposition $X = X_0 + M + A$, where M is a local martingale that is zero at 0 and A is a predictable process of finite variation that is zero at 0.

We let $L_0(X)$ denote the space of real, optional processes H such that

$$\int^t |H_s||dA_s| < \infty \text{ a.s. for each } t \tag{2.1}$$

and

$$\left(\int^t H_s^2 d[M, M]_s\right)^{1/2} \text{ is locally integrable.} \tag{2.2}$$

We note the following:

(a) Condition (2.1) means that the process $H{\cdot}A = \int^t H_s dA_s$ is well-defined, up to an evanescent set.

(b) In particular, if H is locally bounded, then H is in $L_0(X)$. For let (T_n) be a sequence of stopping times which increase to infinity such that $|H| 1_{]0,T_n]}$ is bounded, by a finite constant c_n, for each n. Since M, and hence, $[M, M]$, is zero at 0, we have

$$\left(\int^{T_n} H_s^2 d[M, M]_s\right)^{1/2} = \left(\int_{]0,T_n]} H_s^2 d[M, M]_s\right)^{1/2}$$
$$\leq c_n [M, M]_{T_n}^{1/2} .$$

By Theorem II.4.5, we may also assume $[M, M]_{T_n}^{1/2}$ is integrable; hence, condition (2.2) is satisfied.

Since $A_0 = 0$, we have

$$\int^{T_n} |H_s||dA_s| = \int_{]0,T_n]} |H_s||dA_s|$$

$$\leq c_n |A_{T_n}|.$$

Condition (2.1) then holds, since (T_n) increase to infinity.

3 THEOREM. Let X be a special semimartingale, having canonical decomposition $X = X_0 + M + A$. For every H in $L_0(X)$ there exists a unique semimartingale Z, with decomposition $Z = H_0X_0 + W + H \cdot A$, such that for every bounded E-valued martingale Y, the process $[Z, Y] - H \cdot [X, Y]$ is a real local martingale which is zero at 0.

Z is called the optional (or compensated) stochastic integral of H with respect to X and is denoted by $Z = H \underset{c}{\cdot} X$.

If H is predictable, we write $Z = H \cdot X$ and in this case we in fact have $[H \cdot X, Y] = H \cdot [X,Y]$ for every bounded martingale Y. Moreover, if H is locally bounded and predictable, $H \cdot X$ agrees with the predictable stochastic integral.

Proof: We begin with uniqueness. We must show the uniqueness of the local martingale W in the decomposition of Z. For every bounded martingale Y,

$$(3.1) \qquad [Z, Y] - H \cdot [X, Y] = (H_0X_0, Y_0) + [W, Y] + \sum_{s \leq t} (H_s \Delta A_s, \Delta Y_s)$$

$$- [H_0(X_0, Y_0) + H\cdot[M, Y] + \sum_{s\leq t} H_s(\Delta A_s, \Delta Y_s)]$$

$$= [W, Y] - H\cdot[M, Y].$$

If U is another local martingale, which is zero at 0, that may be used in the decomposition of Z, then for every bounded martingale Y, the processes $[W, Y] - H\cdot[M, Y]$, $[U, Y] - H\cdot[M, Y]$, and, by taking differences, $[W-U, Y]$ are all local martingales which are zero at 0.

We replace W - U by V in the notation. V is a local martingale which is zero at 0 and which we claim is identically zero. Let (T_n) be a sequence of stopping times reducing V. Since V^{T_n} is a uniformly integrable martingale, we know by Theorem 8.7 in Métivier and Pellaumail [10] that V^{T_n} is of class D for each n.

By Theorem II.3.5, the process

(3.2) $$(V^{T_n}, Y) - [V^{T_n}, Y]$$

is a local martingale which is zero at 0. So is the process [V, Y], and then, by stopping at T_n, so is the process

(3.3) $$[V, Y]^{T_n} = [V^{T_n}, Y].$$

By adding the right hand side of (3.3) to (3.2), we have that for every n,

(3.4) $$(V^{T_n}, Y)$$

is a local martingale which is zero at 0, for every bounded martingale Y.

Since V^{T_n} is of class D and Y is bounded and $|(V^{T_n}, Y)| \leq |V^{T_n}||Y|$, we have that (V^{T_n}, Y) is also of class D and hence is a true uniformly integrable martingale [4, VI.30(f)], which is zero at 0.

Thus, for every bounded martingale Y

$$E[(V_{T_n}, Y_\infty)] = 0.$$

In other words, for every bounded F_{T_n}-measurable Y_∞, we have

$$E[(V_{T_n}, Y_\infty)] = 0;$$

hence, $V_{T_n} = 0$ a.s. for each n.

Thus, $V_t^{T_n} = E(V_{T_n} | F_t) = 0$ and V is zero on $[0, T_n]$. Letting n tend to infinity, we have that V is identically 0. The uniqueness is established.

To establish existence, it suffices, in view of (3.1), to construct a local martingale W, which is zero at 0, such that for every bounded martingale Y the process $[W, Y] - H\cdot[M, Y]$ is a local martingale which is zero at 0. To achieve this goal, we need condition (2.2) of Definition 2. In case H is predictable, again in view of (3.1), we need only show $[W, Y] = H\cdot[M, Y]$.

We proceed in several steps.

Step 1: Assume H is bounded and M is in H^1. By Theorem II.5.3, there exists a sequence of bounded, and hence square integrable, martingales (X^n) converging to M in H^1. Referring to the proof of II.5.3, we see that we may take $X_0^n = 0$ for each n, since $M_0 = 0$.

We let $W^n = H\underset{c}{\cdot}X^n$. W^n is in $M_E^2 = H^2$, which is contained in H^1, for each n. Moreover, by Theorem 1.5 (a), $W^n - W^m = H\underset{c}{\cdot}(X^n - X^m)$. We choose a constant c such that $|H| \leq c$, then, by applying Lemma 1, we obtain

$$\begin{aligned}\|W^n - W^m\|_{H^1} &= \|H\underset{c}{\cdot}(X^n - X^m)\|_{H^1} \\ &\leq 3\sqrt{2}\ E[(\int^{\infty} H_s^2 d[X^n - X^m, X^n - X^m]_s)^{1/2}] \\ &\leq 3c\sqrt{2}\ E[[X^n - X^m, X^n - X^m]_{\infty}^{1/2}] \\ &= 3c\sqrt{2}\ \|X^n - X^m\|_{H^1}.\end{aligned}$$

Thus, (W^n) is Cauchy in H^1. Since H^1 is complete (see Appendix), there exists a local martingale W such that (W^n) converges to W in H^1.

Since $X_0^n = 0$ for each n, $W_0^n = H_0 X_0^n = 0$. It is immediate that $((W^n)^0)$ converges to W^0 in H^1 and hence (W_0^n) converges to W_0 in L^1 (see again Theorem II.5.3). Hence, $W_0 = 0$.

For every Y in BMO^2, in particular for every bounded martingale, we have for every stopping time T

$$E[|\int^T d[W, Y]_s - \int^T H_s d[M, Y]_s - \int^T d[W^n, Y]_s + \int^T H_s d[X^n, Y]_s|]$$

$$\le E[\int^{\infty}|d[W-W^n, Y]_s|] + E[\int^{\infty}|H_s||d[M-X^n, Y]_s|]$$

$$\le \sqrt{2}\ \|W - W^n\|_{H^1}\|Y\|_{BMO^2} + c\sqrt{2}\ \|M - X^n\|_{H^1}\|Y\|_{BMO^2}$$

by Corollary II.5.7. Hence,

$$L^1 - \lim_n(\int^T d[W^n, Y]_s - \int^T H_s d[X^n, Y]_s) \tag{3.5}$$

$$= \int^T d[W, Y]_s - \int^T H_s d[M, Y]_s.$$

By Theorem 1.4 (c), $[W^n, Y] - H\bullet[X^n, Y]$ is a uniformly integrable martingale, which is zero at 0. Hence, for every stopping time T,

$$E[\int^T d[W, Y]_s - \int^T H_s d[M, Y]_s]$$

$$= \lim_n E[\int^T d[W^n, Y]_s - \int^T H_s d[X^n, Y]_s]$$

$$= 0.$$

Hence, $[W, Y] - H\bullet[M, Y]$ is also a uniformly integrable martingale. It is clearly zero at 0, since $W_0 = 0$ and $M_0 = 0$. We may then denote $W = H\underset{c}{\bullet}M$.

We note that if H is predictable, then by Theorem 1.4 (c), $[W^n, Y] = H\bullet[X^n, Y]$ for each n. Hence by (3.5) we have for every stopping time T that

$$\int^T d[W, Y]_s - \int^T H_s d[M, Y]_s = 0 \text{ a.s.}$$

Hence, $[W, Y] - H\bullet[M, Y] = 0$ by the cross-section theorem.

Step 2: We assume M is in H^1 and

$$E[(\int^\infty H_s^2 d[M, M]_s)^{1/2}] < \infty.$$

We then have that $\int^\infty H_s^2 d[M, M]_s < \infty$ a.s. and we henceforth consider only w such that $\int^\infty H_s^2 d[M, M]_s < \infty$, and we denote by Γ the set of such w.

We let $H^n = H1_{\{|H| \le n\}}$.

Since H^n converges pointwise to H and $|H^n| \le H$ we have on Γ, by dominated convergence, $\lim \int^\infty (H_s - H_s^n)^2 d[M, M]_s = 0$. Thus, $\lim_n (\int^\infty (H_s - H_s^n)^2 d[M, M]_s)^{1/2} = 0$ a.s. Since,

$$(\int^\infty (H_s - H_s^n)^2 d[M, M]_s)^{1/2} = (\int^\infty (H_s 1_{\{|H|>n\}})^2 d[M, M]_s)^{1/2}$$
$$\le (\int^\infty H_s^2 d[M, M]_s)^{1/2},$$

we have again by dominated convergence that

$$(3.6) \qquad \lim_n E[(\int^\infty (H_s - H_s^n)^2 d[M, M]_s)^{1/2}] = 0.$$

Moreover, it immediately follows that

$$(3.7) \qquad \lim_{n,m} E[(\int^\infty (H_s^n - H_s^m)^2 d[M, M]_s)^{1/2}] = 0.$$

By Step 1, we define $W^n = H^n \underset{c}{\cdot} X$. Then W^n is a local martingale which is zero at 0 for each n. Moreover, for every Y in BMO^2, the process

$$[W^n, Y] - H^n \cdot [M, Y]$$

is a uniformly integrable martingale, which is zero at 0, and is identically zero if H is predictable. Hence,

$$(3.8) \qquad E[[W^n, Y]_\infty - \int^\infty H_s^n d[M, Y]_s] = 0.$$

However, by Theorem II.5.6,

$$\begin{aligned} E[\int^\infty |H_s^n| |d[M, Y]|_s] &\leq \sqrt{2}\ \|Y\|_{BMO^2} E[(\int^\infty (H_s^n)^2 d[M, M]_s)^{1/2}] \\ &\leq \sqrt{2}\ \|Y\|_{BMO^2} E[(\int^\infty H_s^2 d[M, M]_s)^{1/2}] \\ &< \infty. \end{aligned}$$

That is, $\int^\infty H_s^n d[M, Y]_s$ is integrable and (3.8) becomes

$$(3.9) \qquad E[[W^n, Y]_\infty] = E[\int^\infty H_s^n d[M, Y]_s].$$

We may use (3.9) in line (1.1) of the proof of Lemma 1 and thus extend the lemma in general to

$$(3.10) \qquad \|H^n \underset{c}{\cdot} M\|_{H^1} \leq 3\sqrt{2}\ E[(\int^\infty (H_s^n)^2 d[M, M]_s)^{1/2}]$$

for M in H^1 and for any bounded, optional H^n. Moreover, since for every bounded martingale Y

$$\begin{aligned} &[H^n \underset{c}{\cdot} M - H^m \underset{c}{\cdot} M, Y] - (H^n - H^m)\cdot[M, Y] \\ &\qquad = [H^n \underset{c}{\cdot} M, Y] - H^n \cdot [M, Y] - ([H^m \underset{c}{\cdot} M, Y] - H^m \cdot [M, Y]) \end{aligned}$$

is a local martingale which is zero at 0, we have by uniqueness that

$$(3.11) \qquad W^n - W^m = H^n \underset{c}{\cdot} M - H^m \underset{c}{\cdot} M = (H^n - H^m) \underset{c}{\cdot} M.$$

Finally, since

$$\|W^n - W^m\|_{H^1} = \|(H^n - H^m) \underset{c}{\cdot} M\|_{H^1}$$

$$\leq 3\sqrt{2}\ E[(\int^\infty (H^n_s - H^m_s)^2 d[M, M]_s)^{1/2}],$$

(W^n) is Cauchy in H^1, by (3.7).

Hence, there exists a local martingale W in H^1 such that (W^n) converges to W. As in Step 1, we have $W_0 = 0$.

Let Y be in BMO^2. For every stopping time T,

$$E[|\int^T d[W, Y]_s - \int^T H_s d[M, Y]_s - \int^T d[W^n, Y]_s - \int^T H^n_s d[M, Y]_s|]$$

$$\leq E[\int^\infty |d[W-W^n, Y]_s|] + E[\int^\infty |H_s - H^n_s||d[M, Y]_s|]$$

$$\leq \sqrt{2}\ \|Y\|_{BMO^2} \|W - W^n\|_{H^1} + \sqrt{2}\ \|Y\|_{BMO^2} E[(\int^\infty (H_s - H^n_s)^2 d[M, M]_s)^{1/2}]$$

by Corollary II.5.7 and Theorem. II.5.6. Hence,

$$L^1 - \lim_n (\int^T d[W^n, Y]_s - \int^T H^n_s d[M, Y]_s) = \int^T d[W, Y]_s - \int^T H_s d[M, Y]_s.$$

We conclude as in Step 1 that for every Y in BMO^2

$$(3.12) \qquad [W, Y] - H \cdot [M, Y]$$

is a uniformly integrable martingale, which is zero at 0, and which is identically zero if H is predictable. We may now denote $W = H \underset{c}{\cdot} X$.

Step 3: We simply assume condition (2.2) of Definition 2 holds.

We let (T_n) be an increasing sequence of sstopping times such that

$$\left(\int^{T_n} H_s^2 d[M, M]_s\right)^{1/2}$$

is integrable for each n. By Theorem II.4.5, we may also assume that

$$[M, M]_{T_n}^{1/2}$$

is integrable for each n. But, $[M, M]_{T_n}^{1/2} = [M^{T_n}, M^{T_n}]_\infty^{1/2}$; hence, M^{T_n} is in H^1 for each n. Moreover,

$$\left(\int^{T_n} H_s^2 d[M, M]_s\right)^{1/2} = \left(\int^\infty H_s^2 d[M^{T_n}, M^{T_n}]_s\right)^{1/2};$$

thus, the conditions of Step 2 are satisfied. We may then define $W^n = H_{\dot{c}}M^{T_n}$, a local martingale which is zero at 0 and satisfies (3.12). Let Y be in BMO^2. Then let

$$A^n = [W^n, Y] - H\cdot[M^{T_n}, Y]. \tag{3.13}$$

A^{n+1} is a uniformly integrable martingale for each n, thus so is $(A^{n+1})^{T_n}$. But

$$\begin{aligned}(A^{n+1})^{T_n} &= [(W^{n+1})^{T_n}, Y] - H\cdot[(M^{T_{n+1}})^{T_n}, Y] \\ &= [(W^{n+1})^{T_n}, Y] - H\cdot[M^{T_n}, Y].\end{aligned}$$

Since Y is arbitrary, we have by uniqueness that

$$(W^{n+1})^{T_n} = W^n.$$

So we may define $W = \lim_n W^n$ by pasting together, so that for each n

$$W^{T_n} = W^n$$

is a uniformly integrable martingale that is zero at 0. That is, W is a local martingale that is zero at 0.

Letting $A = [W, Y] - H\cdot[M, Y]$, we see that (T_n) also reduce A by (3.13). Hence, A is a local martingale which is zero at 0, for every Y in BMO^2. Moreover, by stopping at T_n and applying (3.12), we see that $A^{T_n} = A^n$ is identically zero if H is predictable. Thus, so is A. We then set $W = H\underset{c}{\cdot}M$ and the construction is complete.

Finally, if H is locally bounded and predictable we know by Theorem II.4.6 that for every bounded martingale Y $[H\cdot X, Y] - H\cdot[X, Y] = 0$. That is, the predictable stochastic integral satisfies the theorem; hence, by uniqueness $H\underset{c}{\cdot}X = H\cdot X$.

We give comments on the theorem.

(a) As was shown throughout the proof, we have in fact that for every Y in BMO^2, the process $[H\underset{c}{\cdot}X, Y] - H\cdot[X, Y]$ is a local martingale which is zero at 0 and is identically zero if H is predictable.

(b) We shall commonly write $H \underset{c}{\cdot} X = H_0 X_0 + H \underset{c}{\cdot} M + H \cdot A$, where $X = X_0 + M + A$ is the canonical decomposition of X, and we note that $(H \underset{c}{\cdot} M)_0 = 0 = (H \cdot A)_0$.

(c) In the real-valued case, $H \underset{c}{\cdot} M$ is the "classical" compensated integral for local martingales given in Déllacherie and Meyer [4]. We will prove a result analogous to Corollary 1.3 and then will be able to use the results in Déllacherie and Meyer [4] for real-valued M, and then extend them to E-valued M as in Theorem 1.7, for example.

(d) Finally, we note that, by (3.1), we have

$$[H \underset{c}{\cdot} X, Y] - H \cdot [X, Y] = [H \underset{c}{\cdot} M, Y] - H \cdot [M, Y]$$

for every bounded martingale Y (even for every semimartingale Y); thus, properties of $H \underset{c}{\cdot} X$ will often reduce to properties of $H \underset{c}{\cdot} M$.

4 COROLLARY. Let X be a special semimartingale and let H be in $L_0(X)$. Then for every f in E, $(f, H \underset{c}{\cdot} X) = H \underset{c}{\cdot} (f, X)$.

Proof: By Theorem II.4.9 (b), H is in $L_0((f, X))$; hence, $H \underset{c}{\cdot} (f, X)$ exists.

Let Y be a bounded (real) martingale. Then by Theorem II.3.9,

$$[(f, X), Y] = [X, fY]$$

and

$$[(f, H\underset{c}{\cdot}X), Y] = [H\underset{c}{\cdot}X, fY].$$

Hence, since fY is an E-valued bounded martingale,

$$[(f, H\underset{c}{\cdot}X), Y] - H\cdot[(f, X), Y] = [H\underset{c}{\cdot}X, fY] - H\cdot[X, fY]$$

is a local martingale which is zero at 0. Thus, by uniqueness of $H\underset{c}{\cdot}(f, X)$, we have that

$$(f, H\underset{c}{\cdot}X) = H\underset{c}{\cdot}(f, X).$$

4. Properties of the Optional Integral

We continue this chapter with a section giving the properties of the optional integral defined in Section 3. In certain cases, we shall restrict ourselves to locally bounded H. We shall also use analogous results for real-valued X, by applying Corollary 3.4.

1 THEOREM.

(a) If H is in $L_0(X)$ and $L_0(Z)$, then H is in $L_0(X+Z)$ and $H\underset{c}{\cdot}(X+Z) = H\underset{c}{\cdot}X + H\underset{c}{\cdot}Z$.

(b) If H and K are in $L_0(X)$, then H + K is in $L_0(X)$ and $(H + K)\underset{c}{\cdot}X = H\underset{c}{\cdot}X + K\underset{c}{\cdot}X$.

Proof:

(a) Let $X = X_0 + M + A$ and $Z = Z_0 + N + B$ be canonical decompositions of the special semimartingales X and Z. Then X + Z has the canonical decomposition.

$$X + Z = (X_0 + Z_0) + (M + N) + (A + B).$$

We have

$$\int^t |H_s|\,|d(A + B)_s| \leq \int^t |H_s|\,|dA_s| + \int^t |H_s|\,|dB_s|,$$

for every t. Also,

$$\int^t H_s^2 d[M+N,\ M+N]_s = \int^t H_s^2 d[M,\ M]_s + 2\int^t H_s^2 d[M,\ N]_s$$
$$+ \int^t H_s^2 d[N,\ N]_s.$$

For the second summand on the right hand side we have,

$$\int^t |H_s|\,|H_s|\,d[M,\ N]_s \leq \int^t |H_s|\,|H_s|\,|d[M,\ N]_s|$$
$$\leq \left(\int^t H_s^2 d[M,\ M]_s\right)^{1/2} \left(\int^t H_s^2 d[N,\ N]_s\right)^{1/2}$$

by the Kunita-Watanabe inequality. Hence,

$$\left(\int^t H_s^2 d[M+N,\ M+N]_s\right)^{1/2}$$
$$\leq \left[\int^t H_s^2 d[M,\ M]_s + 2\left(\int^t H_s^2 d[M,\ M]\ \right)^{1/2} \left(\int^t H_s^2 d[N,\ N]_s\right)^{1/2} + \int^t H_s^2 d[N,\ N]_s\right]^{1/2}$$
$$= \left(\int^t H_s^2 d[M,\ M]_s\right)^{1/2} + \left(\int^t H_s^2 d[N,\ N]_s\right)^{1/2}.$$

Hence, conditions (2.1) and (2.2) of Definition 3.2 are satisfied. That is, H is in $L_0(X+Z)$. Then for every bounded martingale Y,

$$[H_{\overset{\bullet}{c}}X + H_{\overset{\bullet}{c}}Z,\ Y] - H\cdot[X+Z,\ Y]$$
$$= ([H_{\overset{\bullet}{c}}X,\ Y] - H\cdot[X,\ Y]) + ([H_{\overset{\bullet}{c}}Z,\ Y] - H\cdot[Z,\ Y])$$

is a local martingale which is zero at 0. Thus, by uniqueness, $H \underset{c}{\cdot} X + H \underset{c}{\cdot} Z = H \cdot (X + Z)$.

(b) Using the inequalities

(i) $(a + b)^2 \leq 2a^2 + 2b^2$ and

(ii) $(a + b)^{1/2} \leq a^{1/2} + b^{1/2}$, for $a \geq 0$ and $b \geq 0$,

we have for $X = X_0 + M + A$,

$$\begin{aligned}
&\left(\int^t |H_s + K_s|^2 d[M, M]_s\right)^{1/2} \\
&\leq \left(2\int^t |H_s|^2 d[M, M]_s + 2\int^t |K_s|^2 d[M, M]_s\right)^{1/2} \\
&\leq \sqrt{2} \left(\int^t |H_s|^2 d[M, M]_s\right)^{1/2} + \sqrt{2}\left(\int^t |K_s|^2 d[M, M]_s\right)^{1/2}.
\end{aligned}$$

Then since

$$\int^t |H_s + K_s||dA_s| \leq \int^t |H_s||dA_s| + \int^t |K_s||dA_s|,$$

we have that the conditions of Definition 3.2 are satisfied for $H + K$. Hence, $H + K$ is in $L_0(X)$. Then for every bounded martingale Y,

$$\begin{aligned}
&[H \underset{c}{\cdot} X + K \underset{c}{\cdot} X, Y] - (H + K) \cdot [X, Y] \\
&\qquad = [H \underset{c}{\cdot} X, Y] - H \cdot [X, Y] + [K \underset{c}{\cdot} X, Y] - K \cdot [X, Y]
\end{aligned}$$

is a local martingale which is zero at 0. By uniqueness,

$$H \underset{c}{\cdot} X + K \underset{c}{\cdot} X = (H + K) \underset{c}{\cdot} X.$$

2 THEOREM. Let H be in $L_0(X)$. Then for every stopping time T, $(H\underset{c}{\cdot}X)^T = H\underset{c}{\cdot}X^T = H1_{[0,T]}\underset{c}{\cdot}X$.

Proof: If X has canonical decomposition $X = X_0 + M + A$, we note that A^T is still a predictable process and X^T then is special with canonical decomposition $X^T = X_0 + M^T + A^T$. We have

$$(2.1) \qquad \int^t |H_s||dA^T_s| \le \int^t |H_s||dA_s|, \text{ for each } t,$$

$$(2.2) \qquad \int^t |H_s|1_{[0,T]}|dA_s| \le \int^t |H_s||dA_s|, \text{ for each } t,$$

$$(2.3) \qquad \left(\int^t H_s^2 d[M^T, M^T]_s\right)^{1/2} = \left(\int^t H_s^2 d[M, M]^T_s\right)^{1/2}$$

$$\le \left(\int^t H_s^2 d[M, M]_s\right)^{1/2}, \quad \text{and}$$

$$(2.4) \qquad \left(\int^t H_s^2 1_{[0,T]} d[M, M]_s\right)^{1/2} \le \left(\int^t H_s^2 d[M, M]_s\right)^{1/2},$$

which means H is in $L_0(X^T)$ and $H1_{[0,T]}$ is in $L_0(X)$.

Let Y be a bounded martingale and let $B = [H\underset{c}{\cdot}X, Y] - H\cdot[X, Y]$. Then

$$[(H\underset{c}{\cdot}X)^T, Y] - H\cdot[X^T, Y] = [H\underset{c}{\cdot}X, Y]^T - H\cdot[X, Y]^T = B^T$$

is a local martingale which is zero at 0, since B is such. Hence, $(H\underset{c}{\cdot}X)^T = H\underset{c}{\cdot}X^T$. Moreover,

$$[(H\underset{c}{\cdot}X)^T, Y] - H1_{[0,T]}\cdot[X, Y] = [H\underset{c}{\cdot}X, Y] - (H\cdot[X, Y])^T = B^T;$$

hence, $(H\underset{c}{\cdot}X)^T = H1_{[0,T]}\underset{c}{\cdot}X$.

3 THEOREM. Let H be locally bounded, $X = X_0 + M + A$ a special semimartingale, and let $Z = H_{\overset{\bullet}{c}}X$. Then, $Z_0 = H_0X_0$ and $\Delta Z = H\Delta M - {}^p(H\Delta M) + H\Delta A$ on $]0, \infty[$.

Proof: Since $Z = H_0X_0 + H_{\overset{\bullet}{c}}M + H\cdot A$, with $(H_{\overset{\bullet}{c}}M)_0 = 0 = (H\cdot A)_0$, the first assertion is obvious. Since $\Delta(H\cdot A) = H\Delta A$, it suffices to show that $\Delta(H_{\overset{\bullet}{c}}M) = H\Delta M - {}^p(H\Delta M)$ on $]0, \infty[$. But since the result is known for real-valued M, we may prove the result weakly, by Corollary 3.4, provided ${}^p(H\Delta M)$ is suitably defined.

Since H is locally bounded, we know H is in $L_0(X)$ (remark (b) after Definition 3.2). Let (T_n) be an increasing sequence of stopping times such that

$$\left(\int^{T_n} H_s^2 d[M, M]_s\right)^{1/2} = \left(\int^\infty H_s^2 d[M^{T_n}, M^{T_n}]_s\right)^{1/2}$$

is integrable. By Theorem II.4.5, we may also assume that

$$[M^{T_n}, M^{T_n}]_\infty^{1/2} = [M, M]_{T_n}^{1/2}$$

is integrable. Then for every predictable stopping time T,

$$E[{}^p|H\Delta M^{T_n}|_T 1_{(T<\infty)}] = E[|H_T\Delta M_T^{T_n}| 1_{(T<\infty)}]$$

$$= E[(\int_{[T]} H_s^2 d[M^{T_n}, M^{T_n}]_s)^{1/2} 1_{(T<\infty)}]$$

$$\leq E[(\int^{\infty} H_s^2 d[M^{T_n}, M^{T_n}]_s)^{1/2}]$$

$$< \infty.$$

By the predictable cross-section theorem, ${}^p|H\Delta M^{T_n}| < \infty$, outside of an evanescent set. Hence, ${}^p(H\Delta M^{T_n})$ exists [see 2].

Since $1_{[0,T_n]}$ is predictable and

$$1_{[0,T_n]}H\Delta M = 1_{[0,T_n]}H\Delta M^{T_n},$$

we have

$$(3.1) \qquad {}^p(1_{[0,T_n]}H\Delta M) = 1_{[0,T_n]}{}^p(H\Delta M^{T_n}).$$

Moreover,

$$\begin{aligned} 1_{[0,T_n]}{}^p(1_{[0,T_{n+1}]}H\Delta M) &= {}^p(1_{[0,T_n]}1_{[0,T_{n+1}]}H\Delta M) \\ &= {}^p(1_{[0,T_n]}H\Delta M). \end{aligned}$$

Thus, we may paste together to define

$${}^p(H\Delta M) = \lim_n {}^p(1_{[0,T_n]}H\Delta M),$$

so that on $[0, T_n]$, ${}^p(H\Delta M)$ exists classically and

$$1_{[0,T_n]}{}^p(H\Delta M) = {}^p(1_{[0,T_n]}H\Delta M).$$

It follows that for every f in E,

$$(f, {}^p(H\Delta M)) = {}^p((f, H\Delta M))$$

by localizing, applying the result to ${}^p(1_{[0,T_n]}H\Delta M)$ which holds classically, then passing to the limit.

Hence, for every f in E, we have outside of an evanescent set that, on $]0, \infty[$,

$$\begin{aligned}(f, \Delta(H\underset{c}{\cdot}M) &= \Delta H\underset{c}{\cdot}(f, M) \\ &= H\Delta(f, M) - {}^p(H\Delta(f, M)) \\ &= (f, H\Delta M) - {}^p((f, H\Delta M)) \\ &= (f, H\Delta M - {}^p(H\Delta M)).\end{aligned}$$

Hence, in the separable space E, $\Delta(H\underset{c}{\cdot}M)$ and $H\Delta M - {}^p(H\Delta M)$ are indistinguishable on $]0, \infty[$, and the theorem is proved.

(a) We note that the result actually holds for H in $L_0(X)$. In the construction of $H\underset{c}{\cdot}M$ during the proof of Theorem 3.3, one may show at each step that $\Delta(H\underset{c}{\cdot}M) = H\Delta M - {}^p(H\Delta M)$ on $]0, \infty[$. This property will still hold on passing to H^1 limits.

(b) If H is predictable, then ${}^{p}(H\Delta M) = H\,{}^{p}(\Delta M) = 0$, since ${}^{p}M = M_{-}$. Thus, we obtain $\Delta(H\underset{c}{\cdot}X) = H\Delta X$ on $[0, \infty[$.

4 THEOREM. Let H be locally bounded and let K be a locally bounded, predictable, real-valued process. Then $H\underset{c}{\cdot}(K\cdot X) = HK\underset{c}{\cdot}X = K\cdot(H\underset{c}{\cdot}X)$.

Proof: We write $X = X_0 + M + A$, the canonical decomposition. Since K is predictable, $K\cdot A$ is also predictable and hence $K\cdot X = K_0X_0 + K\cdot M + K\cdot A$ is a special semimartingale.

Since H, K, and hence HK, are locally bounded, all the integrals exist. Then

$$H\underset{c}{\cdot}(K\cdot X) = H_0(K_0X_0) + H\underset{c}{\cdot}(K\cdot M) + H\cdot(K\cdot A),$$

$$HK\underset{c}{\cdot}X = (H_0K_0)X_0 + HK\underset{c}{\cdot}M + HK\cdot A,$$

$$K\cdot(H\underset{c}{\cdot}X) = K_0(H_0X_0) + K\cdot(H\underset{c}{\cdot}M) + K\cdot(H\cdot A).$$

In each case, the third summand is the usual Stieltjes integral; hence, $H\cdot(K\cdot A) = HK\cdot A = K\cdot(H\cdot A)$.

Thus, it suffices to prove the result for M. But then we may apply Corollary 3.4, since the result is known for real-valued M. The details are as in Theorem 1.7.

5 COROLLARY. Let $X = X_0 + M$ be a quasi-left-continuous E-valued local martingale. If H and K are both real, locally bounded, optional

processes, then

$$H\underset{c}{\cdot}(K\underset{c}{\cdot}X) = HK\underset{c}{\cdot}X = K\underset{c}{\cdot}(H\underset{c}{\cdot}X).$$

Proof: As in Theorem 4, the problem reduces to showing the result for M. But we may again apply Corollary 3.4, for the result is known for real-valued M.

We note that, in general, if $X = X_0 + M + A$ is a special semimartingale, then $K\underset{c}{\cdot}X$ is a semimartingale, though not necessarily special. Hence, $H\underset{c}{\cdot}(K\underset{c}{\cdot}X)$ is not defined in general which explains the restraint on X in Corollary 5. We will see after Theorem 8 why the optional integral is not defined for a general semimartingale.

6 THEOREM. Let $X = X_0 + M + A$ be a special semimartingale such that M is quasi-left-continuous.

(a) For every locally bounded H, $\Delta(H\underset{c}{\cdot}X) = H\Delta X$.

(b) If H is locally bounded, then for every E-valued semimartingale Y, $[H\underset{c}{\cdot}X, Y] = H\cdot[X, Y]$.

Proof:

(a) As in the proof of Theorem 3, it suffices to show $\Delta(H\underset{c}{\cdot}M) = H\Delta M$. This follows immediately from Corollary 3.4, since the result is known for real-valued M.

(b) We have noted that for every semimartingale Y,

$$[H\underset{c}{\cdot}X, Y] - H\cdot[X, Y] = [H\underset{c}{\cdot}M, Y] - H\cdot[M, Y].$$

We decompose Y as $Y = N + B$, where N is a local martingale that is zero at 0 and has bounded jumps, and B is a process of finite variation. Then,

$$\begin{aligned}&[H\underset{c}{\cdot}M, Y] - H\cdot[M, Y]\\ &\quad = [H\underset{c}{\cdot}M, N] - H\cdot[M, N] + \sum_{s\leq t} (\Delta(H\underset{c}{\cdot}M)_s, \Delta B_s)\\ &\quad\quad - \sum_{s\leq t} H_s(\Delta M_s, \Delta B_s).\end{aligned}$$

By part (a),

$$\sum_{s\leq t} (\Delta(H\underset{c}{\cdot}M)_s, \Delta B_s) = \sum_{s\leq t} H_s(\Delta M_s, \Delta B_s).$$

Hence, we must show that $[H\underset{c}{\cdot}M, N] - H\cdot[M, N] = 0$.

Let (T_n) be a sequence of stopping times which increase to infinity such that N^{T_n} is bounded by n, for each n. Then N^{T_n} is a bounded local martingale and hence is a true bounded martingale.

Let

$$\begin{aligned}U^n &= ([H\underset{c}{\cdot}M, N] - H\cdot[M, N])^{T_n}\\ &= [H\underset{c}{\cdot}M, N^{T_n}] - H\cdot[M, N^{T_n}].\end{aligned}$$

U^n is a local martingale which is zero at 0, by Theorem 3.3, and U^n is a continuous process of finite variation, since by part (a)

$$\begin{aligned}\Delta[H_c\cdot M, N^{T_n}] &= (\Delta(H_c\cdot M), \Delta N^{T_n}) \\ &= H(\Delta M, \Delta N^{T_n}) \\ &= \Delta(H\cdot[M, N^{T_n}]).\end{aligned}$$

We assert U^n must be identically 0. Let (S_m) be a sequence of stopping times reducing the local martingale U^n. Then $V^m = (U^n)^{S_m}$ is a uniformly integrable martingale which is also a continuous process of finite variation. Then for every stopping time R,

$$E(V^m_\infty - V^m_{R-}|F_R) = V^m_R - V^m_{R-} = 0.$$

Thus, V^m and 0 have the same optional dual projection [4, VI.75]. But since V^m is optional, we have $(U^n)^{S_m} = V^m = 0$. Since S_m increases to infinity, we have $([H_c\cdot M, N] - H\cdot[M, N])^{T_n} = U^n = 0$, for each n. It follows that $[H_c\cdot M, N] - H\cdot[M, N] = 0$ and the theorem is proved.

In general, Theorem 6 (b) will not hold for arbitrary semimartingales; however, we can give a slight generalization of Theorem 3.3.

7 THEOREM. Let $X = X_0 + M + A$ be a special semimartingale and let H be bounded and optional. Then for every local martingale Y, $[H_c\cdot X, Y] - H\cdot[X, Y]$ is a local martingale which is zero at 0.

Proof: Once again, we are reduced to showing the result for $[H\underset{c}{\cdot}M, Y] - H\cdot[M, Y]$. We may then assume that $Y_0 = 0$, for

$$\begin{aligned}
&[H\underset{c}{\cdot}M, Y] - H\cdot[M, Y] \\
&\quad = [H\underset{c}{\cdot}M, Y-Y_0] - H\cdot[M, Y-Y_0] + ((H\underset{c}{\cdot}M)_0, Y_0) - H_0(M_0, Y_0) \\
&\quad = [H\underset{c}{\cdot}M, Y-Y_0] - H\cdot[M, Y-Y_0] + (0, Y_0) - H_0(0, Y_0) \\
&\quad = [H\underset{c}{\cdot}M, Y-Y_0] - H\cdot[M, Y-Y_0].
\end{aligned}$$

We write $Y = N + P$, where N is a local martingale with bounded jumps, P is a local martingale which is also a process of finite variation, and $N_0 = 0 = P_0$.

As in the proof of 6 (b), we let (T_n) be a sequence of stopping times which increase to infinity such that N^{T_n} is bounded for each n. We again have that $([H\underset{c}{\cdot}M, N] - H\cdot[M, N])^{T_n}$ is a local martingale which is zero at 0, for each n. Thus, the process $[H\underset{c}{\cdot}M, N] - H\cdot[M, N]$ is also a local martingale which is zero at 0 [4, VI.30].

Thus, we have left to show the result for

$$[H\underset{c}{\cdot}M, P] - H\cdot[M, P].$$

Since P is of finite variation and $P_0 = 0$ we have,

$$\begin{aligned}
&[H\underset{c}{\cdot}M, P] - H\cdot[M, P] \\
&\quad = \sum_{0<s\leq t} (\Delta(H\underset{c}{\cdot}M)_s, \Delta P_s) - \sum_{0<s\leq t} H_s(\Delta M_s, \Delta P_s)
\end{aligned}$$

$$= \sum_{0<s\leq t} (H_s\Delta M_s, \Delta P_s) - \sum_{0<s\leq t} ({}^p(H\Delta M)_s, \Delta P_s) - \sum_{0<s\leq t} (H_s\Delta M_s, \Delta P_s)$$

$$= \sum_{0<s\leq t} ({}^p(H\Delta M)_s, \Delta P_s).$$

$$= \sum_{0\leq s\leq t} ({}^p(H\Delta M)_s, \Delta P_s).$$

We now decompose $M = W + Z$, where W is a local martingale with bounded jumps, Z is a local martingale of finite variation, and $W_0 = 0 = Z_0$. Then,

$$\sum_{0\leq s\leq t} ({}^p(H\Delta M)_s, \Delta P_s) = \sum_{s\leq t} ({}^p(H\Delta W)_s, \Delta P_s) + \sum_{s\leq t} ({}^p(H\Delta Z)_s, \Delta P_s). \tag{7.2}$$

Since H is bounded and ΔW is bounded and thin (that is, ΔW is zero outside of a countable union of disjoint graphs of stopping times), $H\Delta W$ is bounded and thin. Thus, ${}^p(H\Delta W)$ is also bounded. Moreover, $H\Delta W - {}^p(H\Delta W)$ is also thin, by extending the result that $X - {}^pX$ is thin for any real, bounded, optional X [4, VI.46] to the separable Hilbert space E. Hence, ${}^p(H\Delta W)$ must also be thin.

P is both a local martingale and a process of finite variation; thus, the predictable stochastic integral ${}^p(H\Delta W)\cdot P$ is a local martingale, which is zero at 0, since $P_0 = 0$, and is also the usual Stieltjes integral. Then since ${}^p(H\Delta W)$ is thin, we have

$$^{p}(H\Delta W)\cdot P = \int^{t} {}^{p}(H\Delta W)_s dP_s$$

$$= \sum_{0\leq s\leq t} ({}^{p}(H\Delta W)_s, \Delta P_s).$$

is a local martingale which is zero at 0.

For the second summand in (7.2), we note that Z is a local martingale, zero at 0, that is also a process of finite variation. Hence, Z is locally integrable. That is, the process $\int^t |dZ_s|$ is locally integrable. But then $\int^t |H_s||dZ_s|$ is locally integrable, since H is bounded. Hence, $(H\cdot Z)^p$ exists and is a predictable process of finite variation whose jumps are given by

$$\Delta(H\cdot Z)^p = {}^{p}(\Delta(H\cdot Z)) = {}^{p}(H\Delta Z).$$

Then since P is a local martingale, we have by Theorem I.3.4 that

$$\sum_{s\leq t} (\Delta(H\cdot Z)^p_s, \Delta P_s) = \sum_{s\leq t} ({}^{p}(H\Delta Z)_s, \Delta P_s)$$

is a local martingale, which is zero at 0, since $P_0 = 0$.

We now come to the relation of the optional integral to the Stieltjes integral. It is a matter of determining $H\underset{c}{\cdot}M$, where $X = X_0 + M + A$, in the case M is also a process of finite variation, since we already know that $H\cdot A$ is the Stieltjes integral.

8 THEOREM. Suppose M is an E-valued local martingale, zero at 0, that is also a process of finite variation and appears in the canonical decomposition of a special semimartingale X. If H is in

$L_0(X)$ and $\int^t |H_s||dM_s| < \infty$ a.s. for each t, then $\int^t |H_s||dM_s|$ is locally integrable and $H\underset{c}{\cdot}M = B - B^p$, where $B_t = \int^t H_s dM_s$.

Proof: Let $C_t = (\int^t H_s^2 d[M, M]_s)^{1/2}$. C is locally integrable, since H is in $L_0(X)$. Let $D_t = \int^t |H_s||dM_s|$. Then

$$\begin{aligned} D_t - D_{t-} = \Delta D_t &= |H_t||\Delta M_t| \\ &= (H_t^2 \Delta[M, M]_t)^{1/2} \\ &\leq (\int^t H_s^2 d[M, M]_s)^{1/2}. \end{aligned}$$

Hence, $D_t \leq D_{t-} + C_t$ and D is locally integrable, since C is and since D_- is increasing and left continuous, hence locally integrable.

By Lemma 2.2, for every bounded martingale Y

$$[B - B^p, Y] - H\cdot[M, Y]$$

is a local martingale that is zero at 0. Hence, $H\underset{c}{\cdot}M = B - B^p$.

Theorem 8 explains why we cannot define an extension of the predictable stochastic integral to optional processes for arbitrary semimartingales.

Let $X = X_0 + M + A$ be a decomposition of the semimartingale X and let H be a bounded, optional process. One would like to define

$$H\underset{c}{\cdot}X = H_0X_0 + H\underset{c}{\cdot}M + H\cdot A$$

and then show the independence of the choice of decompositions of X.

However, if $X = X_0 + N + B$ is another decomposition, then by Theorem 8

$$H\underset{c}{\bullet}(M - N) = H\bullet(B - A) - (H\bullet(B - A))^p.$$

Hence,

$$(H\underset{c}{\bullet}M + H\bullet A) - (H\underset{c}{\bullet}N + H\bullet B) = -(H\bullet(B - A))^p$$

is nonzero in general. However, if H is predictable, then

$$(H\bullet(B - A))^p = H\bullet(B - A)^p = 0,$$

since $B - A$ is a local martingale of finite variation and thus $(B - A)^p = 0$, and there is not a problem.

In the case of a special semimartingale X, which has a unique decomposition $X = X_0 + M + A$, where A is a predictable process of finite variation, there is not a problem. We may separately define

$$\text{(iii)} \qquad H\underset{c}{\bullet}A = H\bullet A = \int^t H_s dA_s,$$

and we have agreement with the predictable integral when H is predictable. Since A is predictable, it cannot be a local martingale without being identically 0; thus, Theorem 8 offers no ambiguity.

We shall conclude this section with two convergence theorems analogous to Theorem 1.6. We refer the reader to the Appendix for the definitions of S^p-spaces [4, VII.98]. We begin with a lemma.

9 LEMMA. Let M be an E-valued local martingale which is zero at 0 and let H be in $L_0(M)$. Then $\|H{\underset{c}{\cdot}}M\|_{H^1} \leq 3\sqrt{2}\, E[(\int^\infty H_s^2 d[M, M]_s)^{1/2}]$.

Proof: Let (T_n) be a sequence of stopping times which increase to infinity such that

$$[M^{T_n}, M^{T_n}]_\infty^{1/2} = [M, M]_{T_n}^{1/2}$$

is integrable (that is, M^{T_n} is in H^1) and

$$(\int^\infty H_s^2 d[M^{T_n}, M^{T_n}]_s)^{1/2} = (\int^{T_n} H_s^2 d[M, M]_s)^{1/2}$$

is integrable, for each n.

By (3.12) in Step 2 of the proof of Theorem 3.3, we have for every Y in BMO^2 that

$$E[[H{\underset{c}{\cdot}}M^{T_n}, Y]_\infty] = E[\int^\infty H_s d[M^{T_n}, Y]_s].$$

Thus, letting $W = H{\underset{c}{\cdot}}M$ and $W^n = (H{\underset{c}{\cdot}}M)^{T_n} = H{\underset{c}{\cdot}}M^{T_n}$, we obtain

$$E[[W, W]_{T_n}^{1/2}] = E[[W^n, W^n]_\infty^{1/2}]$$

$$= \|W^n\|_{H^1}$$

$$= \sup_{\substack{Y \text{ in } BMO^2 \\ \|Y\| \leq 3}} |E[[W^n, Y]_\infty]|$$

$$\leq \sup_{\substack{Y \text{ in } BMO^2 \\ \|Y\| \leq 3}} E[\int^\infty |H_s| \, |d[M^{T_n}, Y]_s|$$

$$\leq \sup_{\substack{Y \text{ in } BMO^2 \\ \|Y\| \leq 3}} \sqrt{2} \, \|Y\|_{BMO^2} E[(\int^\infty H_s^2 d[M^{T_n}, M^{T_n}]_s)^{1/2}]$$

$$\leq 3\sqrt{2} \, E[(\int^{T_n} H_s^2 d[M, M]_s)^{1/2}].$$

We let n tend to infinity and apply the monotone convergence theorem and obtain

$$\|W\|_{H^1} = E[[W, W]_\infty^{1/2}]$$
$$\leq 3\sqrt{2} \, E[(\int^\infty H_s^2 d[M, M]_s)^{1/2}].$$

10 THEOREM. Let $X = X_0 + M + A$ be a special semimartingale. Let (H^k) be a uniformly bounded sequence of optional processes which converge pointwise to H. Then $(H^k \underset{c}{\cdot} X)$ converges locally in S^1 to $H \underset{c}{\cdot} X$.

Proof: We have $H^k \underset{c}{\cdot} X = H_0^k X_0 + H^k \underset{c}{\cdot} M + H^k \cdot A$ and $H \underset{c}{\cdot} X = H_0 X_0 + H \underset{c}{\cdot} M + H \cdot A$. The conclusion of the theorem means that there exists a sequence of stopping times (T_n) which increase to infinity such that for each n, $(H^k \underset{c}{\cdot} X)^{T_n} 1_{(T_n > 0)}$ is in S^1 for each k, $(H \underset{c}{\cdot} X)^{T_n} 1_{(T_n > 0)}$ is in

S^1 and

$$S^1 = \lim_k (H^k_c \cdot X)^{T_n} 1_{(T_n > 0)} = (H_c \cdot X)^{T_n} 1_{(T_n > 0)}.$$

We note that

$$\begin{aligned}(H_c \cdot X)^{T_n} 1_{(T_n > 0)} &= H_0 X_0 1_{(T_n > 0)} + (H_c \cdot M)^{T_n} + (H_c \cdot A)^{T_n} \\ &= H_0 X_0 1_{(T_n > 0)} + H_c \cdot M^{T_n} + H_c \cdot A^{T_n}.\end{aligned}$$

We have a similar formula for each H^k.

We choose (T_n) as follows:

(i) $Y = X_0 + M$ is a local martingale. We let (T_n) be a sequence of times reducing Y. Then Y_0 is integrable on $(T_n > 0)$. That is, $X_0 1_{(T_n > 0)}$ is integrable for each n.

(ii) By Theorem II.4.5, $[M, M]_t^{1/2}$ is locally integrable; thus, we may assume

$$E[[M^{T_n}, M^{T_n}]_\infty^{1/2}] = E[[M, M]_{T_n}^{1/2}] < \infty$$

for each n.

(iii) A is a predictable process of finite variation; thus, A is also locally integrable. Hence, we may further assume that

$$E[\int^{\infty}|dA_s^{T_n}|] = E[\int^{T_n}|dA_s|] < \infty$$

for each n.

Now we fix n and write $T = T_n$. We have that each H^k, and hence H, is bounded by the same constant c. Hence,

$$(10.1) \quad \|(H\underset{c}{\cdot}X)^T 1_{(T>0)}\|_{S^1}$$

$$\leq \|H_0X_0 1_{(T>0)}\|_{L^1} + \|H\underset{c}{\cdot}M^T\|_{H^1} + E[\int^{\infty}|H_s||dA_s^T|]$$

$$\leq c\|X_0 1_{(T>0)}\|_{L^1} + 3\sqrt{2}\, E[(\int^{\infty}H_s^2 d[M^T, M^T]_s)^{1/2}] + cE[\int^T|dA_s|]$$

$$\leq c\|X_0 1_{(T>0)}\|_{L^1} + 3\sqrt{2}\, cE[[M, M]_T^{1/2}] + cE[\int^T|dA_s|]$$

$$< \infty.$$

Hence, $(H\underset{c}{\cdot}X)^T 1_{(T>0)}$ is in S^1. Likewise, $(H^k\underset{c}{\cdot}X)^T 1_{(T>0)}$ is in S^1 for each K. Since $(H^k\underset{c}{\cdot}X)^T 1_{(T>0)} - (H\underset{c}{\cdot}X)^T 1_{(T>0)} = ((H^k - H)\underset{c}{\cdot}X)^T 1_{(T>0)}$ we may replace H by $H^k - H$ in (10.1) and obtain

$$\lim_k \|(H^k\underset{c}{\cdot}X)^T 1_{(T>0)} - (H\underset{c}{\cdot}X)^T 1_{(T>0)}\|_{S^1}$$

$$\leq \lim_k(\|(H_0^k - H_0)X_0 1_{(T>0)}\|_{L^1} + 3\sqrt{2}\, E[(\int^{\infty}(H_s^k - H_s)^2 d[M^T, M^T]_s)^{1/2}]$$

$$+ E[\int^{\infty}|H^k_s - H_s||dA^T_s|])$$

$$= 0,$$

by the dominated convergence theorem.

11 THEOREM. Let (X^k) be a sequence of special semimartingales converging to a special semimartingale X in S^1. If H is a bounded, optional process, then $H\underset{c}{\cdot}X^k$ converges to $H\underset{c}{\cdot}X$ in S^1.

Proof: We write canonical decompositions, $X^k = X^k_0 + M^k + A^k$ and $X = X_0 + M + A$.

Since X^k converges to X in S^1, we have

(i) X^k_0 converges to X_0 in L^1.

(ii) M^k converges to M in H^1.

(iii) $\lim\limits_k E[\int^{\infty}|d(A^k_s - A_s)|] = 0.$

Then, choosing c such that $|H| \leq c$, we have

$$\|H\underset{c}{\cdot}X^k - H\underset{c}{\cdot}X\|_{S^1} = \|H\underset{c}{\cdot}(X^k - X)\|_{S^1}$$

$$\leq \|H_0(X^k_0 - X_0)\|_{L^1} + \|H\underset{c}{\cdot}(M^k - M)\|_{H^1}$$

$$+ E[\int^{\infty}|H_s||d(A^k_s - A_s)|]$$

$$\leq c\|(X^k_0 - X_0)\|_{L^1} + 3\sqrt{2}\, E[(\int^{\infty}H^2_s d[M^k-M, M^k-M]_s)^{1/2}]$$

$$+ cE[\int^{\infty}|d(A_s^k - A_s)|]$$

$$\leq c\|(X_0^k - X_0)\|_{L^1} + 3\sqrt{2}\, c\|M^k - M\|_{H^1}$$

$$+ cE[\int^{\infty}|d(A_s^k - A_s)|].$$

It simply remains to let k tend to infinity.

5. Decomposition of Local Martingales

We conclude this chapter with a section applying the optional integral to the decomposition of E-valued local martingales (see the remarks of II.2.9).

1 DEFINITION. An E-valued local martingale X is called purely discontinuous if the process $[X, X]_t$ is indistinguishable from $\sum_{s\leq t} |\Delta X_s|^2$. X is called continuous if X has continuous paths and $X_0 = 0$.

2 THEOREM. If X is purely discontinuous, then $[X, Y]_t = \sum_{s\leq t} (\Delta X_s, \Delta Y_s)$, for every semimartingale Y.

Proof: We have

$$[X, Y]_t = \int^t d[X, Y]_s$$

$$= \int^t 1_{\{\Delta X_s=0\}} d[X, Y]_s + \int^t 1_{\{\Delta X_s\neq 0\}} d[X, Y]_s.$$

By the Kunita-Watanabe inequality, we have for every t

$$\begin{aligned}\left|\int^t 1_{\{\Delta X_s = 0\}} d[X, Y]_s\right| &\leq \int^t 1_{\{\Delta X_s = 0\}} |d[X, Y]_s| \\ &\leq \left(\int^t 1_{\{\Delta X_s = 0\}} d[X, X]_s\right)^{1/2} \left(\int^t d[Y, Y]_s\right)^{1/2} \\ &= \left(\sum_{s \leq t} 1_{\{\Delta X_s = 0\}} |\Delta X_s|^2\right)^{1/2} [Y, Y]_t^{1/2} \\ &= 0.\end{aligned}$$

By right continuity, $\int^t 1_{\{\Delta X_s = 0\}} d[X, Y]_s$ is indistinguishable from zero. Hence,

$$[X, Y]_t = \int^t 1_{\{\Delta X_s \neq 0\}} d[X, Y]_s.$$

However, since $1_{\{\Delta X_s \neq 0\}}$ is a thin process, the integral reduces to a sum of jumps; hence,

$$\begin{aligned}[X, Y]_t &= \sum_{s \leq t} 1_{\{\Delta X_s \neq 0\}} \Delta [X, Y]_s \\ &= \sum_{s \leq t} 1_{\{\Delta X_s \neq 0\}} (\Delta X_s, \Delta Y_s) \\ &= \sum_{s \leq t} (\Delta X_s, \Delta Y_s).\end{aligned}$$

3 COROLLARY. If X and Y are purely discontinuous, then so is X + Y.

Proof: We have for every t

$$\begin{aligned}[X+Y, X+Y]_t &= [X, X]_t + 2[X, Y]_t + [Y, Y]_t \\ &= \sum_{s\le t} |\Delta X_s|^2 + 2 \sum_{s\le t} (\Delta X_s, \Delta Y_s) + \sum_{s\le t} |\Delta Y_s|^2 \\ &= \sum_{s\le t} (\Delta X_s+\Delta Y_s, \Delta X_s+\Delta Y_s) \\ &= \sum_{s\le t} |\Delta(X + Y)_s|^2.\end{aligned}$$

By right continuity, $[X+Y, X+Y]$ and $\sum_{s\le t} |\Delta(X + Y)_s|^2$ are indistinguishable.

4 THEOREM. Let X be a local martingale and let $J = 1_{\{\Delta X\neq 0\}}$. $W = J_{\dot{c}}X$ is a purely discontinuous local martingale, which is in M_E^2 if X is in M_E^2, and $\Delta W = \Delta X$.

Proof: J is bounded and optional; hence, J is in $L_0(X)$, where the special semimartingale X has canonical decomposition $X = X_0 + (X - X_0)$. Thus, $J_{\dot{c}}X = J_0X_0 + J_{\dot{c}}(X - X_0)$ is a local martingale. If X is in M_E^2, then J is in $L_0^2(X)$ and $J_{\dot{c}}X$ is a square integrable martingale.

Since X is a local martingale, ${}^pX$ exists and ${}^pX = X_-$; hence, ${}^p(\Delta X) = 0$. Since $J\Delta X = \Delta X$, we have by Theorem 4.3 that

$$\Delta W_0 = W_0 = J_0X_0 = J_0\Delta X_0 = \Delta X_0.$$

On $]0, \infty[$, we have

$$\Delta W = J\Delta(X - X_0) - {}^p(J\Delta(X - X_0))$$

$$= J\Delta X - {}^{p}(J\Delta X)$$

$$= \Delta X - {}^{p}(\Delta X)$$

$$= \Delta X.$$

The last assertion is then proved.

By Theorem 4.7, we have for every local martingale Y that the process $A = [W, Y] - J\cdot[X, Y]$ is a local martingale which is zero at 0. Taking $Y = W$ and using the fact that J is thin, we have that

$$\begin{aligned} A_t &= [W, W]_t - \sum_{s\leq t} J_s\Delta[X, W]_s \\ &= [W, W]_t - \sum_{s\leq t} J_s(\Delta X_s, \Delta W_s) \\ &= [W, W]_t - \sum_{s\leq t} (\Delta X_s, \Delta W_s) \\ &= [W, W]_t - \sum_{s\leq t} (\Delta W_s, \Delta W_s) \\ &= [W, W]_t^c \end{aligned}$$

is a local martingale which is zero at 0. But $A = [W, W]^c$ is also a continuous, hence predictable, process of finite variation. Thus, $A = 0$. Hence,

$$\begin{aligned} [W, W]_t &= [W, W]_t^c + \sum_{s\leq t} \Delta[W, W]_s \\ &= \sum_{s\leq t} (\Delta W_s, \Delta W_s) \end{aligned}$$

and W is purely discontinuous.

We now give the decomposition theorem of local martingales.

5 THEOREM. Every local martingale X has a unique decomposition $X = X^c + X^d$, where X^c is a continuous local martingale and X^d is a purely discontinuous local martingale. If X is in M_E^2, then X^c and X^d are also in M_E^2.

Proof: Let $J = 1_{\{\Delta X \neq 0\}}$. We let $X^d = J \underset{c}{\cdot} X$ and $X^c = X - X^d$. By Theorem 4, X^d is purely discontinuous and X^c is continuous.

Suppose $X = M + N$ is another decomposition where M is a continuous local martingale and N is purely discontinuous. Then $U = X^c - M = N - X^d$ is a local martingale, zero at 0, that is both continuous and purely discontinuous. Thus,

$$[U, U]_\infty = \sum_{s<\infty} (\Delta U_s, \Delta U_s) = 0.$$

Hence, by Theorem I.3.7 and Corollary I.3.6, U is a square integrable martingale and

$$E[|U_\infty|^2] = E[[U, U]_\infty] = 0.$$

Hence, $U_\infty = 0$ a.s. and U is indistinguishable from 0. The uniqueness is then established.

6 COROLLARY. Let X be an E-valued local martingale. Then for every f in E, $(f, X)^c = (f, X^c)$ and $(f, X)^d = (f, X^d)$.

Proof: (f, X) has a unique decomposition $(f, X) = (f, X)^c + (f, X)^d$. However, $(f, X) = (f, X^c+X^d) = (f, X^c) + (f, X^d)$.

(f, X^c) is continuous, since $(f, X_0^c) = 0$ and $\Delta(f, X^c) = (f, \Delta X^c) = 0$.

In order to show that (f, X^d) is purely discontinuous, it suffices, in view of Theorem 8 to follow, to show that $[(f, X^d), Y]$ is a local martingale, which is zero at 0, for every real continuous local martingale Y.

If Y is continuous, then fY is an E-valued continuous local martingale. But by Theorem II.3.9,

$$[(f, X^d), Y] = [X^d, fY],$$

which is a local martingale that is zero at 0, again by Theorem 8 to follow.

7 DEFINITION. Local martingales X and Y are called orthogonal if $[X, Y]$ is a local martingale which is zero at 0.

In view of Theorem II.3.5, X and Y are orthogonal if and only if (X, Y) is a local martingale which is zero at 0.

8 THEOREM. X is purely discontinuous if and only if X is orthogonal to every continuous local martingale.

Proof: If X is purely discontinuous, then for every continuous local martingale Y, we obtain by Theorem 2

$$[X, Y]_t = \sum_{s \leq t} (\Delta X_s, \Delta Y_s) = 0 \tag{8.1}$$

Hence, $[X, Y]$ is indistinguishable from 0.

Conversely, we decompose X as $X = X^c + X^d$, according to Theorem 5. Then X is orthogonal to X^c. Hence, $[X, X^c]$ is a local martingale, zero at 0. Since X^d is purely discontinuous, we have by (8.1) that $[X^d, X^c] = 0$. Hence

$$\begin{aligned}[X, X^c] &= [X^c+X^d, X^c] \\ &= [X^c, X^c].\end{aligned}$$

Moreover, $\Delta[X^c, X^c] = |\Delta X^c|^2 = 0$; thus, $[X^c, X^c]$ is a continuous, hence predictable, process of finite variation that is also a local martingale which is zero at 0. Hence, $[X^c, X^c] = 0$. Then by Theorem II.3.7, X^c is a square integrable martingale and

$$E[|X^c_\infty|^2] = E[[X^c, X^c]_\infty] = 0.$$

Hence, $X^c_\infty = 0$ a.s. and $X^c = 0$.

Thus, $X = X^d$ is purely discontinuous.

9 THEOREM. Let X and Y be local martingales. Then

(a) $[X, Y]^d = [X^d, Y^d]$, and

(b) $[X, Y]^c = [X^c, Y^c]$.

Proof:

(a) By Theorem 2, we have

$$[X^d, Y^d]_t = \sum_{s \le t} (\Delta X^d_s, \Delta Y^d_s)$$

$$= \sum_{s \le t} (\Delta X_s, \Delta Y_s)$$

$$= \sum_{s \le t} \Delta [X, Y]_s$$

$$= [X, Y]_t^d.$$

(b) By (8.1), we have $[X^d, Y^c] = 0$ and hence

$$\begin{aligned}
[X^c, Y^c]_t &= [X - X^d, Y^c]_t \\
&= [X, Y^c]_t \\
&= [X, Y]_t - [X, Y^d]_t \\
&= [X, Y]_t - \sum_{s \le t} (\Delta X_s, \Delta Y_s^d) \\
&= [X, Y]_t - \sum_{s \le t} (\Delta X_s, \Delta Y_s) \\
&= [X, Y]_t - [X, Y]_t^d \\
&= [X, Y]_t^c.
\end{aligned}$$

We point out that this result gives an alternate way of defining the square bracket (see the discussion preceding Theorem II.4.8). If X and Y are square integrable martingales, then so are X^c and Y^c; hence, $\langle X^c, Y^c \rangle$ exists. Since $[X^c, Y^c] = [X, Y]^c$ is continuous, we have $[X^c, Y^c]$ is then predictable and $[X^c, Y^c] = \langle X^c, Y^c \rangle$. Thus,

$$\begin{aligned}
[X, Y]_t &= [X, Y]_t^c + [X, Y]_t^d \\
&= \langle X^c, Y^c \rangle_t + \sum_{s \le t} (\Delta X_s, \Delta Y_s)
\end{aligned}$$

is an expression often taken as the definition of the square bracket.

We conclude by giving the decomposition of the optional integral.

10 THEOREM. Let X be a local martingale (square integrable martingale) and let H be in $L_0(X)$ (in $L_0^2(X)$). Then $(H\underset{c}{\cdot}X)^c = H\underset{c}{\cdot}X^c$ and $(H\underset{c}{\cdot}X)^d = H\underset{c}{\cdot}X^d$.

Proof: We note that we may assume $X_0 = 0$ since

$$H\underset{c}{\cdot}X = H_0X_0 + H\underset{c}{\cdot}(X - X_0).$$

Then since

$$\left(\int^t H_s^2 d[X^c, X^c]_s + \int^t H_s^2 d[X^d, X^d]_s\right) = \left(\int^t H_s^2 d[X, X]_s^c + \int^t H_s^2 d[X, X]_s^d\right)$$
$$= \int^t H_s^2 d[X, X]_s,$$

we have that H is in $L_0(X^c)$ and $L_0(X^d)$ (resp. H is in $L_0^2(X^c)$ and $L_0^2(X^d)$).

Moreover, $H\underset{c}{\cdot}X = H\underset{c}{\cdot}(X^c + X^d) = H\underset{c}{\cdot}X^c + H\underset{c}{\cdot}X^d$. We have $(H\underset{c}{\cdot}X^c)_0 = H_0X_0^c = 0$ and on $]0, \infty[$

$$\Delta(H\underset{c}{\cdot}X^c) = H\Delta X^c - {}^p(H\Delta X^c) = 0;$$

hence, $H\underset{c}{\cdot}X^c$ is continuous. We must now show that $H\underset{c}{\cdot}X^d$ is purely discontinuous.

Let Y be a continuous local martingale. then Y is locally bounded. There exists stopping times (T_n) which increase to infinity such that Y^{T_n} is bounded for each n. Y^{T_n} is then a true martingale for each n and hence

$$[H\underset{c}{\cdot}X^d, Y^{T_n}] - H\cdot[X^d, Y^{T_n}]$$

is a local martingale which is zero at 0, by Theorem 3.3.

However, $[X^d, Y^{T_n}]_t = \sum_{s\leq t} (\Delta X^d_s, \Delta Y^{T_n}_s) = 0$, since X^d is purely discontinuous and Y^{T_n} is continuous. Hence, $[H\underset{c}{\cdot}X^d, Y]^{T_n} = [H\underset{c}{\cdot}X^d, Y^{T_n}]$ is a local martingale, zero at 0, for each n. Since (T_n) increase to infinity, $[H\underset{c}{\cdot}X^d, Y]$ is also a local martingale, zero at 0. Thus, $H\underset{c}{\cdot}X^d$ is orthogonal to every continuous local martingale and is purely discontinuous, by Theorem 8.

Since $H\underset{c}{\cdot}X = (H\underset{c}{\cdot}X)^c + (H\underset{c}{\cdot}X)^d$ is a unique decomposition, we have $(H\underset{c}{\cdot}X)^c = H\underset{c}{\cdot}X^c$ and $(H\underset{c}{\cdot}X)^d = H\underset{c}{\cdot}X^d$.

CHAPTER IV
THE STOCHASTIC INTEGRAL OF AN OPTIONAL HILBERT PROCESS

In this chapter, we define and characterize the stochastic integral of an optional Hilbert-valued process H with respect to a real special semimartingale X. As in Chapter III, this integral is an extension of the predictable stochastic integral. We again start by defining the integral with respect to a square integrable martingale and then naturally extend to special semimartingales. The class of optional processes for which we make this extension is slightly smaller than the analogous extension in Chapter III. We shall restrict ourselves here to locally bounded processes. Chapter V will offer an alternate construction of this stochastic integral. Unless otherwise specified, throughout this chapter H will be E-valued and optional and X will be real-valued.

1. The Optional Integral With Respect to a Square Integral Martingale

We begin by defining an extension of the predictable stochastic integral, with respect to a square integrable martingale, X, to optional processes.

1 DEFINITION. Let $L_0^2(X)$ be the space of optional E-valued processes H such that

$$\|H\|_X = (E[\int^\infty |H_s|^2 d[X, X]_s])^{1/2} < \infty.$$

Since $\langle X, X\rangle$ - $[X, X]^p$, if follows that if H is predictable, then H is in $L^2_p(X)$.

2 THEOREM. Let H be in $L^2_0(X)$. There exists a unique E-valued square integrable martingale $H \underset{c}{\cdot} X$, called the optional (or compensated) stochastic integral of H with respect to X, such that for every Y_∞ in $L^2(\Omega, F, P, R)$

$$E[(H \underset{c}{\cdot} X)_\infty Y_\infty] = E[\int^\infty H_s d[X, Y]_s],$$

where Y is a cadlag version of the real square integrable martingale $E(Y_\infty | F_t)$.

Proof: Let (e_n) be an orthonormal basis of E. Then

(2.1) $$H_s = \sum_n (H_s, e_n)e_n$$

and by Parseval's inequality we have for every n

(2.2) $$|(H_s, e_n)|^2 \leq \sum_i |(H_s, e_i)|^2 = |H_s|^2.$$

Hence, $H^n = (H, e_n)$ is a real, optional process and $E[\int^\infty (H^n_s)^2 d[X, X]_s] < \infty$.

Hence, the classical $H^n \underset{c}{\cdot} X$ exists [4, VIII.32], and for every Y_∞ in $L^2(\Omega, F, P, R)$ we have

(2.3) $$E[(H^n \underset{c}{\cdot} X)_\infty Y_\infty] = E[\int^\infty H^n_s d[X, Y]_s].$$

Let $W^n = H^n_c{\cdot}X$. We would like to define $H_c{\cdot}X$ by $H_c{\cdot}X = \sum_n W^n e_n$; however, we must first verify that the sum converges, which is equivalent to showing that $\sum_n (W^n)^2 < \infty$.

For every stopping time T,

$$
\begin{aligned}
(2.4)\qquad E[\sum_n (W^n_T)^2] &= \sum_n E[(W^n_T)^2] \\
&= \sum_n E[[W^n, W^n]_T] \\
&\leq \sum_n E[[W^n, W^n]_\infty] \\
&\leq \sum_n E[\int^\infty (H^n_s)^2 d[X, X]_s] \\
&= E[\int^\infty \sum_n (H^n_s)^2 d[X, X]_s] \\
&= E[\int^\infty |H_s|^2 d[X, X]_s] \\
&< \infty.
\end{aligned}
$$

Thus, outside of an evanescent set, $\sum_n (W^n)^2$ is finite and we may define $W = \sum_n W^n e_n$. In particular, $W_\infty = \sum_n W^n_\infty e_n$ is defined outside of a null set and W_∞ is square integrable, by (2.4), since $|W_\infty|^2 = \sum_n |W^n_\infty|^2$. Furthermore, $(W_\infty, e_n) = W^n_\infty$ for all n.

Let $H_c{\cdot}X$ be a cadlag version of the square integrable martingale $E(W_\infty | F_t)$.

Let Y_∞ be in $L^2(\Omega, F, P, R)$. By the Kunita-Watanabe inequality and Holder's inequality,

$$(2.5) \qquad E[\int^\infty |H_s||d[X, Y]_s|] \leq \|H\|_X \|Y_\infty\|_{L^2};$$

hence, $E[\int^\infty H_s d[X, Y]_s]$ is a well-defined element in E. We then have

$$\begin{aligned} E[\int^\infty H_s d[X, Y]_s] &= \sum_n (E[\int^\infty H_s d[X, Y]_s], e_n)e_n \\ &= \sum_n E[\int^\infty (H_s, e_n) d[X, Y]_s]e_n \\ &= \sum_n E[W^n_\infty Y_\infty]e_n \\ &= \sum_n E[(W_\infty, e_n)Y_\infty]e_n \\ &= \sum_n E[(W_\infty Y_\infty, e_n)]e_n \\ &= \sum_n (E[W_\infty Y_\infty], e_n)e_n \\ &= E[W_\infty Y_\infty] \\ &= E[(H_c \cdot X)_\infty Y_\infty]. \end{aligned}$$

If U is another square integrable martingale satisfying the theorem, then for every Y_∞ in $L^2(\Omega, F, P, R)$,

$$E[U_\infty Y_\infty] = E[(H_c \cdot X)_\infty Y_\infty].$$

Hence, for every Y_∞ in $L^2(\Omega, F, P, E)$, we may write

$$Y_\infty = \sum_n Y^n e_n$$

where $Y^n = (Y_\infty, e_n)$ is in $L^2(\Omega, F, P, R)$ for each n, since $|(Y_\infty, e_n)|^2 \leq \sum_i |(Y_\infty, e_i|^2 = |Y_\infty|^2$.

Write $U_\infty = \sum_n (U_\infty, e_n)e_n = \sum_n U^n e_n$. Then,

$$(U_\infty, Y_\infty) = (U_\infty, \sum_n Y^n e_n) = \sum_n (U_\infty Y^n, e_n),$$

and we have pointwise that

$$\lim_k \sum_{n=1}^{k} (U_\infty Y^n, e_n) = (U_\infty, Y_\infty).$$

However, the limit holds in L^1 by dominated convergence, since for every k

$$\begin{aligned} |\sum_{n=1}^{k} (U_\infty Y^n, e_n)| &\leq \sum_{n=1}^{\infty} |(U_\infty Y^n, e_n)| \\ &= \sum_{n=1}^{\infty} |(U_\infty, e_n)Y^n| \\ &= \sum_{n=1}^{\infty} |U^n||Y^n| \\ &\leq (\sum_{n=1}^{\infty} |U^n|^2)^{1/2}(\sum_{n=1}^{\infty} |Y^n|^2)^{1/2} \\ &= |U_\infty||Y_\infty| \in L^1. \end{aligned}$$

Likewise, we have in L^1 that

$$((H_c\cdot X)_\infty, Y_\infty) = \lim_k \sum_{n=1}^{\infty} ((H_c\cdot X)_\infty Y^n, e_n).$$

Hence,

$$\begin{aligned} E[(U_\infty, Y_\infty)] &= \lim_k E[\sum_{n=1}^{k} (U_\infty Y^n, e_n)] \\ &= \lim_k \sum_{n=1}^{k} (E[U_\infty Y^n], e_n) \\ &= \lim_k \sum_{n=1}^{k} (E[(H_c\cdot X)_\infty Y^n], e_n) \\ &= \lim_k E[\sum_{n=1}^{k} ((H_c\cdot X)_\infty Y^n, e_n)] \\ &= E[((H_c\cdot X)_\infty, Y_\infty)]. \end{aligned}$$

Which gives us $U_\infty = (H_c\cdot X)_\infty$ a.s., and then that U and $H_c\cdot X$ are indistinguishable.

We point out that the theorem may be worded to say $E[(H_c\cdot X)_\infty Y_\infty] = E[\int^\infty H_s d[X, Y]_s]$, for every real square integrable martingale Y, since L^2 and M^2 are isometrically isomorphic.

The result of Theorem 2 merits further examination for its gives an example of a compact integral operator from $L^2(\Omega, F, P, R)$ into E. Let $T(Y_\infty) = E[\int^\infty H_s d[X, Y]_s]$, for every Y_∞ in $L^2(\Omega, F, P, R)$.

T is clearly linear and, by (2.5),

$$|T(Y_\infty)| \leq E[\int^\infty |H_s||d[X, Y]_s|]$$
$$\leq \|H\|_X \|Y_\infty\|_{L^2};$$

hence, T is a bounded, linear operator from $L^2(\Omega, F, P, R)$ into E.

In Theorem 3.1.2, we defined an analogous bounded, linear operator J from $L^2(\Omega, F, P, E)$ into R given by the same expression:

$$J(Y_\infty) = E[\int^\infty H_s d[X, Y]_s,$$

where in this case, H is real-valued and X is E-valued.

We know J has an integral representation since the dual of $L^2(\Omega, F, P, E)$ is again $L^2(\Omega, F, P, E)$.

There is not, however, an analogous statement for the E-valued operator T, based only on the fact that T is bounded and linear. We need a stronger condition which we now state. Let

(2.6) $$|||T|||_2 = \sup(\sum_{i=1}^{n} |a_i T(1_{E_i})|),$$

where the suprenum is taken over all functions $f = \sum_{i=1}^{n} a_i 1_{E_i}$, where (E_i) are mutually pairwise disjoint, E_i is in F for i = 1,...,n, and $\|f\|_{L^2} \leq 1$.

(2.7) There exists W_∞ in $L^2(\Omega, F, P, E)$ such that $E[Y_\infty W_\infty] = T(Y_\infty)$, for every Y_∞ in $L^2(\Omega, F, P, R)$, if and only if $|||T|||_2 < \infty$. In this case, $\|W_\infty\|_{L^2} = |||T|||_2$.

Verifying directly that $|||T|||_2 < \infty$ is certainly a nontrivial, if at all possible, task; however, in Theorem 2, we have explicitly produced $W_\infty = (H \underset{c}{\cdot} X)_\infty$ which satisfies (2.7). Hence, $|||T|||_2 < \infty$. Moreover, since E has the Radon-Nikodym property, T is compact.

These results may be found in Diestel and Uhl [5, p. 110], who give the theory in general for Banach valued operators on L^p. Definition (2.6) is due to Dinculeanu whose monograph [6] was the first to contain a comprehensive study of the theory of vector measures.

3 COROLLARY. Let H be in $L_0^2(X)$. For every f in E, $(f, H \underset{c}{\cdot} X) = (f,H) \underset{c}{\cdot} X$.

Proof: Let $W = H \underset{c}{\cdot} X$. We note that

$$(E[\int^\infty |(f, H_s)|^2 d[X, X]_s])^{1/2} \leq |f| \, |H|_X < \infty.$$

Hence, (f, H) is in $L_0^2(X)$ (classically) and $(f, H) \underset{c}{\cdot} X$ exists. Denote $U = (f, H) \underset{c}{\cdot} X$.

Then for every Y_∞ in $L^2(\Omega, F, P, R)$

$$\begin{aligned} E[(f, W_\infty)Y_\infty] &= E[(f, W_\infty Y_\infty)] \\ &= (f, E[W_\infty Y_\infty]) \\ &= (f, E[\int^\infty H_s d[X, Y]_s]) \\ &= E[\int^\infty (f, H_s) d[X, Y]_s] \end{aligned}$$

$$= E[[U, Y]_\infty]$$

$$= E[U_\infty Y_\infty].$$

Hence,

$$U = E((f, W_\infty)|F_t)$$

$$= (f, E(W_\infty|F_t))$$

$$= (f, W).$$

4 THEOREM. Let H be in $L_0^2(X)$ and let $W = H\underset{c}{\cdot}X$.

(a) If H is predictable, then $W = H \cdot X$.

(b) $E[[W, W]_\infty] \le E[\int^\infty |H_s|^2 d[X, X]_s]$.

(c) For every real square integrable martingale Y, the process $A = YW - H\cdot[X, Y]$ is an E-valued uniformly integrable martingale which is zero at 0.

(d) $W_0 = H_0X_0$; $\Delta W = H\Delta X - {}^p(H\Delta X)$ on $]0, \infty[$.

(e) $W^T = H\underset{c}{\cdot}X^T = H1_{[0,T]}\underset{c}{\cdot}X$ for every stopping time T.

Proof:

(a) Let f be in E. Then since the result is known for real-valued H, we have

$$(f, W) = (f, H)\underset{c}{\cdot}X = (f, H)\cdot X = (f, H\cdot X),$$

outside of an evanescent set. Letting f range through a countable norming set, we obtain that W and H·X are indistinguishable.

(b) We use the notations of Theorem 2. Let (e_n) be an orthonormal basis of E. Let $H^n = (H, e_n)$. By Corollary 3, we have $(W, e_n) = (H, e_n)\cdot_c X = H^n \cdot_c X$. Let $W^n = (W, e_n)$.

Then $W = \sum_n W^n e_n$ and $|W|^2 = \sum_n |W^n|^2$. Hence, by (2.4)

$$\begin{aligned} E[[W, W]_\infty] &= E[|W_\infty|^2] \\ &= E[\sum_n |W^n_\infty|^2] \\ &\leq E[\int^\infty |H_s|^2 d[X, X]_s]. \end{aligned}$$

(c) Let Y be a real square integrable martingale. For every stopping time T, Y^T is also in M^2; hence, by Theorem 2,

$$\begin{aligned} E[W_\infty Y_T] &= E[W_\infty Y^T_\infty] \\ &= E[\int^\infty H_s d[X, Y^T]_s] \\ &= E[\int^\infty H_s d[X, Y]^T_s] \\ &= E[\int^T H_s d[X, Y]_s]. \end{aligned}$$

On the other hand,

$$E[W_\infty Y^T] = E[E(W_\infty Y_T | F_T)]$$

$$= E[E(W_\infty | F_T) Y_T]$$

$$= E[W_T Y_T].$$

Hence, for every stopping time T,

$$E[W_T Y_T - \int^T H_s d[X, Y]_s] = 0$$

and thus $A = WY - H \cdot [X, Y]$ is a uniformly integrable martingale. To show it is zero at 0, it suffices to show $W_0 = H_0 X_0$, that is (d), since $A_0 = W_0 Y_0 - H_0 X_0 Y_0$.

(d) As in Theorem III.1.4 (d), we may prove this result weakly. Let f be in E; then almost surely $(f, W_0) = (f, H_c^{\cdot} X)_0 = ((f, H)_c^{\cdot} X)_0 = (f, H_0) X_0 = (f, H_0 X_0)$.

Hence, $W_0 = H_0 X_0$ a.s. in the separable Hilbert space E. For every predictable T,

$$E[{}^p|H \Delta X|_T 1_{(T<\infty)}] = E[|H_T \Delta X_T| 1_{(T<\infty)}]$$

$$= E[|H_T| |\Delta X_T|]$$

$$= E[(\int_{[T]} |H_s|^2 d[X, X]_s)^{1/2}]$$

$$\le (E[\int_{[T]} |H_s|^2 d[X, X]_s])^{1/2}$$

$$\le \|H\|_X.$$

Thus, outside of an evanescent set, ${}^p|H \Delta X| < \infty$ and then ${}^p(H \Delta X)$ exists by Brooks and Dinculeanu [2].

Then, outside of an evanescent set, we have for f in E that, on]0, ∞[,

$$\begin{aligned}(f, \Delta W) &= \Delta(f, W)\\ &= \Delta((f, H)\underset{c}{\cdot}X)\\ &= (f, H)\Delta X - {}^{p}((f, H)\Delta X)\\ &= (f, H\Delta X) - {}^{p}((f, H\Delta X))\\ &= (f, H\Delta X) - (f, {}^{p}(H\Delta X))\\ &= (f, H\Delta X - {}^{p}(H\Delta X)).\end{aligned}$$

Hence, ΔW and $H\Delta X - {}^{p}(H\Delta X)$ are indistinguishable on]0, ∞[.

(e) We first note that for every stopping time T,

$$\begin{aligned}E[\int^{\infty}|H_s|^2 d[X^T, X^T]_s] &= E[\int^{\infty}|H_s|^2 d[X, X]_s^T]\\ &= E[\int^{T}|H_s|^2 d[X, X]_s]\\ &\leq E[\int^{\infty}|H_s|^2 d[X, X]_s]\end{aligned}$$

and

$$\begin{aligned}E[\int^{\infty}|H_s 1_{[0,T]}|^2 d[X, X]_s] &= E[\int^{T}|H_s|^2 d[X, X]_s]\\ &\leq E[\int^{\infty}|H_s|^2 d[X, X]_s]\end{aligned}$$

so $H\underset{c}{\cdot}X^T$ and $H1_{[0,T]}\underset{c}{\cdot}X$ exist.

Then for every f in E, up to indistinguishability,

$$\begin{aligned}(f,\ H\underset{c}{\cdot}X^T) &= (f,\ H)\underset{c}{\cdot}X^T\\ &= ((f,\ H)\underset{c}{\cdot}X)^T\\ &= ((f,\ H\underset{c}{\cdot}X))^T\\ &= (f,\ (H\underset{c}{\cdot}X)^T)\end{aligned}$$

and

$$\begin{aligned}(f,\ H\underset{c}{\cdot}X^T) &= (f,\ H)\underset{c}{\cdot}X^T\\ &= (f,\ H)1_{[0,T]}\underset{c}{\cdot}X\\ &= (f,\ H1_{[0,T]})\underset{c}{\cdot}X\\ &= (f,\ H1_{[0,T]}\underset{c}{\cdot}X).\end{aligned}$$

Hence, $H\underset{c}{\cdot}X^T$, $(H\underset{c}{\cdot}X)^T$, and $H1_{[0,T]}\underset{c}{\cdot}X$ are all indistinguishable.

This result may also be proved in a direct manner as in Theorem III.1.4 (e).

5 THEOREM.

(a) If H is in both $L_0^2(X)$ and $L_0^2(Z)$, then H is in $L_0^2(X+Z)$ and $H\underset{c}{\cdot}(X+Z) = H\underset{c}{\cdot}X + H\underset{c}{\cdot}Z$.

(b) If H and K are in $L_0^2(X)$, then H + K is in $L_0^2(X)$ and $(H+K)\underset{c}{\cdot}X = H\underset{c}{\cdot}X + K\underset{c}{\cdot}X$.

Proof:

(a) The same argument as Theorem III.1.5 (a) shows $\|H\|_{X+Z} \leq \|H\|_X + \|H\|_Z$ and hence that H is in $L_0^2(X+Z)$. Replacing inner products by products in the same proof, we have $H\underset{c}{\cdot}(X+Z) = H\underset{c}{\cdot}X + H\underset{c}{\cdot}X$.

(b) As in Theorem III.1.5 (b), there exists a real P-measure m such that for every real measurable process Y,

$$\int Y^2 d_m = E[\int^\infty Y_s^2 d[X, X]_s]. \tag{5.1}$$

In particular, for H in $L_0^2(X)$

$$\int |H_s|^2 dm = E[\int^\infty |H_s|^2 d[X, X]_s]$$

and $L_0^2(X)$ is a subset of $L^2(R^+ \times \Omega, B(R^+) \times F, m, E)$. Again by Minkowski's inequality

$$\|H + K\|_X \leq \|H\|_X + \|K\|_X$$

and H + K is in $L_0^2(X)$.

The linearity follows as in Theorem III.1.5 (b), by replacing inner products by products.

The following theorem is analogous to Theorem III.1.6 with an identical proof, which we thus omit.

6 THEOREM.

(a) Let (H^k) be a sequence of processes in $L_0^2(X)$ which are uniformly bounded in norm by a process L in $L_0^2(X)$. If H^k

converges pointwise to H, then H is in $L_0^2(X)$ and $H^k \underset{c}{\cdot} X$ converges in M_E^2 to $H \underset{c}{\cdot} X$.

(b) Let H be a bounded, optional, E-valued process. If (X^n) is a sequence of real square integrable martingales converging to X in M^2, then $H \underset{c}{\cdot} X^n$ converges to $H \underset{c}{\cdot} X$ in M_E^2.

7 THEOREM.

(a) Let H be in $L_0^2(X)$. If K is a real-valued, predictable process in both $L_p^2(X)$ and $L_p^2(H \underset{c}{\cdot} X)$ and $E[\int^\infty |H_s|^2 |K_s|^2 d[X, X]_s] < \infty$, then $H \underset{c}{\cdot} (K \cdot X) = HK \underset{c}{\cdot} X = K \cdot (H \underset{c}{\cdot} X)$.

(b) The same result is true if we take H to be real-valued and K to be E-valued.

Proof: By hypothesis, $K \cdot X$ and $K \cdot (H \underset{c}{\cdot} X)$ exist. In case (a) and (b), we may apply Theorem II.4.7 and obtain

$$E[\int^\infty |H_s|^2 d[K \cdot X, K \cdot X]_s] = E[\int^\infty |H_s|^2 |K_s|^2 d[X, X]_s] < \infty.$$

Hence, $H \underset{c}{\cdot} (K \cdot X)$ and $HK \underset{c}{\cdot} X$ exist.

(a) We know the result for real H [4, VIII.33]. Let f be in E. Then outside of an evanescent set

$$\begin{aligned}(f, H \underset{c}{\cdot} (K \cdot X)) &= (f, H) \underset{c}{\cdot} (K \cdot X)\\ &= (f, H) K \underset{c}{\cdot} X\\ &= (f, HK) \underset{c}{\cdot} X\end{aligned}$$

$$= (f,\ HK\underset{c}{\bullet}X)$$

and

$$\begin{aligned}(f,\ HK\underset{c}{\bullet}X) &= (f,\ HK)\underset{c}{\bullet}X\\ &= (f,\ H)K\underset{c}{\bullet}X\\ &= K\bullet((f,\ H)\underset{c}{\bullet}X)\\ &= K\bullet(f,\ H\underset{c}{\bullet}X)\\ &= (f,\ K\bullet(H\underset{c}{\bullet}X)) \quad \text{by II.2.6 (b).}\end{aligned}$$

Hence, $H\underset{c}{\bullet}(K\bullet X)$, $HK\underset{c}{\bullet}X$ and $K\bullet(H\underset{c}{\bullet}X)$ are all indistinguishable in E.

The proof of (b) is similar (or see Theorem 4.4).

We close the section with one last application of Corollary 3.

8 THEOREM. Let H be in $L^2_0(X)$ and $W = H\underset{c}{\bullet}X$. If X is quasi-left-continuous, then

(a) $\Delta W = H\Delta X$.

(b) Theorem 7 remains valid for optional K, with the added hypothesis in 7 (b) that H is in $L^2_0(H\underset{c}{\bullet}X)$.

Proof:

(a) Let f be in E. The result is known for real H; hence, outside of an evanescent set

$$(f, \Delta W) = \Delta(f, W) = \Delta(f, H)\underset{c}{\cdot}X = (f, H)\Delta X = (f, H\Delta X).$$

Hence, $\Delta W = H\Delta X$.

(b) We must first verify that the integrals in 7 are still defined for optional K, with the hypotheses obviously changed to read "K is in $L_0^2(X)$ and $L_0^2(H\underset{c}{\cdot}X)$."

In the case of 7 (a), we may apply Theorem III.1.9 (b) twice to obtain

$$[K\underset{c}{\cdot}X, K\underset{c}{\cdot}X] = K^2\cdot[X, X].$$

Hence,

$$E[\int^\infty |H_s|^2 d[K\underset{c}{\cdot}X, K\underset{c}{\cdot}X]_s] = E[\int^\infty |H_s|^2 |K_s|^2 d[X, X]_s]$$

and then H is in $L_0^2(K\underset{c}{\cdot}X)$ and HK is in $L_0^2(X)$.

In the case of 7 (b), we have $H\underset{c}{\cdot}(K\underset{c}{\cdot}X)$ exists, by the added hypothesis. Applying Theorem III.1.9 (b) twice, we obtain

$$[H\underset{c}{\cdot}X, H\underset{c}{\cdot}X] = H^2\cdot[X, X],$$

giving

$$E[\int^\infty |K_s|^2 d[H\underset{c}{\cdot}X, H\underset{c}{\cdot}X]_s] = E[\int^\infty |K_s|^2 |H_s|^2 d[X, X]_s].$$

Hence, K is in $L_0^2(H\underset{c}{\cdot}X)$ (which is no longer needed as a hypothesis) and HK is in $L_0^2(X)$.

Having all the integrals existing, we may apply Corollary 3 as in Theorem 7 to obtain the result.

2. Relation to the Stieltjes Integral

In this section, we obtain results analogous to those in Section III.2 for when X is a square integrable martingale that is also a process of finite variation. Before doing so however, we give some results regarding products of the form MA, where M is real-valued and A is E-valued. These properties will replace previous properties of the square bracket, which is not at our disposal for products of the above kind.

1 THEOREM. Let M be a real local martingale which is zero at 0 and let A be an E-valued predictable process of finite variation. Then,

(a) $MA - \int^t M_{s-} dA_s$ is a local martingale which is zero at 0;

(b) $MA - \int^t M_s dA_s$ is a local martingale which is zero at 0, even without A being predictable.

Proof: The proof is identical to the case when A is real-valued [4, VII.35].

2 COROLLARY. Let M be a real local martingale which is zero at 0 and let A be an E-valued predictable process of finite variation. Then the process $\int^t \Delta M_s dA_s = \sum_{s \leq t} \Delta M_s \Delta A_s$ is a local martingale which is zero at 0.

Proof: Since ΔM is a thin process, that is it is zero outside of a countable union of disjoint graphs of stopping times, the process

$\int^t \Delta M_s dA_s$ reduces to the sum $\sum_{s\leq t} \Delta M_s \Delta A_s$; hence, the equality holds in the statement. The result follows by subtracting the process in 1 (b) from that in 1 (a).

3 COROLLARY. Let M be a real local martingale which is zero at 0 and let N be an E-valued local martingale which is also a process of finite variation. Then the process $MN - \sum_{s\leq t} \Delta M_s \Delta N_s$ is also a local martingale which is zero at 0.

Proof: By Theorem 1 (b), we have

$$MN - \int^t M_s dN_s \tag{3.1}$$

is a local martingale which is zero at 0.

Since M_- is predictable and locally bounded and N is of finite variation, the predictable stochastic integral $M_- \cdot N$ is given by

$$\int^t M_{s-} dN_s \tag{3.2}$$

which is also a local martingale, since N is, and is zero at 0, again since N is. Adding (3.1) and (3.2), we obtain that

$$MN - \int^t \Delta M_s dN_s = MN - \sum_{s\leq t} \Delta M_s \Delta N_s$$

is a local martingale which is zero at 0.

The following result is proven exactly as its analog Lemma III.2.1.

4 LEMMA. Let H be in $L_0^2(X)$. Then $H_{\dot{c}}X = H_{\dot{c}}(X - X_0) + H_0X_0$.

5 LEMMA. Let X be a real local martingale which is zero at 0 and let H be an E-valued optional process. If X is also a process of finite variation and if the process $\int^t |H_s||dX_s|$ is locally integrable, then, letting $A_t = \int^t H_s dX_s$ and $N_t = A_t - A_t^p$, we have for every real local martingale Y that the process $NY - \int^t H_s d[X, Y]_s$ is a local martingale which is zero at 0.

Proof: As in Theorem III.2.2, the process $\int^t |H_s||d[X, Y]_s|$ is finite outside of an evanescent set and hence the process $\int^t H_s d[X, Y]_s$ is defined.

Now since X is of finite variation and $A = N + A^p$, we have

$$\begin{aligned}\int^t H_s d[X, Y]_s &= \sum_{s\le t} H_s \Delta X_s \Delta Y_s \\ &= \sum_{s\le t} \Delta A_s \Delta Y_s \\ &= \sum_{s\le t} \Delta N_s \Delta Y_s + \sum_{s\le t} \Delta A_s^p \Delta Y_s .\end{aligned}$$

Then,

$$NY - \int^t H_s d[X, Y]_s = (NY - \sum_{s\le t} \Delta N_s \Delta Y_s) - \sum_{s\le t} \Delta A_s^p \Delta Y_s .$$

The result follows by applying Corollary 3 to the first term on the right and Corollary 2 to the second term.

6 THEOREM. Let H be in $L_0^2(X)$ where X is also a process of finite variation. If the process $\int^t |H_s||dX_s|$ is finite outside of an evanescent set, then this process is in fact locally integrable and $H_{\dot{c}}X = A - A^p + H_0X_0$, where $A_t = \int^t H_s dX_s$.

Proof: We refer to the proof of the analogous result in Theorem III.2.3 and note that we may again reduce to the case where $X_0 = 0$ and then that $\int^t |H_s||dX_s|$ is locally integrable.

Now let $A_t = \int^t H_s dX_s$ and $N_t = A_t - A_t^p$. As in III.2.3, we have that

$$E[[N, N]_\infty] \leq 4E[\int^\infty |H_s|^2 d[X, X]_s] < \infty$$

and thus that N is a square integrable martingale.

Then for every real square integrable martingale Y, the process $M = NY - \int^t H_s d[X, Y]_s$ is a local martingale which is zero at 0, by Lemma 5.

Let (T_n) be a sequence of stopping times reducing M. Since N and Y are square integrable martingales, we have that N_{T_n} converges to N_∞ in $L^2(E)$ and Y_{T_n} converges to Y_∞ in $L^2(R)$, as n tends to infinity. Hence, $N_{T_n}Y_{T_n}$ converges to $N_\infty Y_\infty$ in $L^1(E)$. Moreover,

$$E[|\int^\infty H_s d[X, Y]_s - \int^{T_n} H_s d[X, Y]_s|]$$

$$= E[|\int^\infty H_s 1_{]T_n,\infty]} d[X, Y]_s|]$$

$$\leq E[\int^{\infty}|H_s|1_{]T_n,\infty]}|d[X, Y]_s|]$$

$$\leq E[(\int^{\infty}|H_s|^2 d[X, X]_s)^{1/2}(\int^{\infty}1_{]T_n,\infty]}d[Y, Y]_s)^{1/2}]$$

$$\leq \|H\|_X(E[\int^{\infty}1_{]T_n,\infty]}d[Y, Y]_s])^{1/2}.$$

By dominated convergence, we have that $E[\int^{\infty}1_{]T_n,\infty]}d[Y, Y]_s]$ converges to 0, as n tends to infinity; hence, $\int^{T_n}H_s d[X, Y]_s$ converges to $\int^{\infty}H_s d[X, Y]_s$ in $L^1(E)$.

Since M^{T_n} is a uniformly integrable martingle which is zero at 0 for each n, we have $E[M_{T_n}] = 0$. That is,

$$E[N_{T_n}Y_{T_n}] = E[\int^{T_n}H_s d[X, Y]_s].$$

Letting n tend to infinity, we have that

$$E[N_{\infty}Y_{\infty}] = E[\int^{\infty}H_s d[X, Y]_s].$$

By Theorem 1.2, we have $N = H_{\dot{c}}X$.

By reducing to the case where $X_0 = 0$, we were actually working with $X - X_0$. We conclude as in Theorem III.2.3: If X is not zero at 0, we obtain $H_{\dot{c}}X = \int^t H_s dX_s - (\int^{\cdot} H_s dX_s)^p + H_0X_0$.

3. The Optional Integral With Respect to a Special Semimartingale

We now extend the notion of the optional integral to integrate an E-valued optional process with respect to a real special semimartingale. The techniques in Chapter III will not apply in this setting; thus, we must restrict ourselves to locally bounded optional processes. In this respect, this extension is much like the predictable stochastic integral in that we shall construct the process by pasting and then piecing together its different parts. As in Chapter III, the essential problem is to define the integral with respect to a local martingale.

1 THEOREM. Let X be a special semimartingale having canonical decomposition $X = X_0 + M + A$, where M is a local martingale, A is a predictable process of finite variation, and $M_0 = 0 = A_0$. Let H be a locally bounded optional process. There exists a unique E-valued semimartingale Z having decomposition $X = H_0X_0 + W + H \cdot A$ such that for every real bounded martingale Y, the process $YW - H \cdot [M, Y]$ is an E-valued local martingale which is zero at 0.

Z is called the optional (or compensated) stochastic integral of H with respect to X and we denote $W = H \underset{c}{\cdot} M$ and $Z = H \underset{c}{\cdot} X$. Moreover, if H is predictable, then $H \underset{c}{\cdot} X = H \cdot X$.

Proof: We begin with uniqueness. Suppose W and V are two local martingales which are zero at 0 such that for every real bounded martingale Y, $YW - H \cdot [M, Y]$ and $YV - H \cdot [M, Y]$ are local martingales which are zero at 0. Then, writing $U = W - V$, we have YU is a local martingale that is zero at 0. We assert $U = 0$.

Let (T_n) be a sequence of stopping times reducing both U and YU. Then for every stopping time S, (1.1) $$E[Y_{S\wedge T_n} U_{S\wedge T_n}] = E[Y_S^{T_n} U_S^{T_n}] = 0.$$

Next note that for every n,

(1.2)
$$\begin{aligned} E[Y_S U_{T_n}] &= E[E(Y_S U_{T_n} | F_{T_n})] \\ &= E[E(Y_S | F_{T_n}) U_{T_n}] \\ &= E[Y_{S\wedge T_n} U_{T_n}] \\ &= E[E(Y_{S\wedge T_n} U_{T_n} | F_S)] \\ &= E[Y_{S\wedge T_n} E(U_{T_n} | F_S)] \\ &= E[Y_{S\wedge T_n} U_{S\wedge T_n}] \\ &= 0. \end{aligned}$$

In particular, if we fix m, then for all n ≥ m,

(1.3) $$E[Y_{T_m} U_{T_n}] = 0.$$

However, for n ≥ m, since $T_n \geq T_m$,

$$\begin{aligned} E[Y_\infty U_{T_m}] &= E[Y_\infty E(U_{T_n} | F_{T_m})] \\ &= E[E(Y_\infty | F_{T_m}) U_{T_n}] \end{aligned}$$

$$= E[Y_{T_m} U_{T_n}]$$

$$= 0.$$

Thus, since Y is arbitrary, we have that $E[Y_\infty U_{T_m}] = 0$, for every bounded random variable Y_∞. In particular

(1.4) $$\int_B U_{T_m}\, dP = \int 1_B U_{T_m}\, dP = 0$$

for every set B in F_{T_m}. Thus, for every f in E,

$$\int_B (f, U_{T_m})dP = (f, \int_B U_{T_m}\, dP) = 0,$$

for every B in F_{T_m}. Hence, $(f, U_{T_m}) = 0$ a.s. for every f in E. Since E is separable, we obtain that $U_{T_m} = 0$ a.s., by letting f range through a countable norming set.

Then, $U^{T_m} = E(U_{T_m} | F_t)$ is indistinguishable from 0. That is, $U = 0$ on $[0, T_m]$. Letting m tend to infinity, we have $U = 0$, and the uniqueness is proved.

A consequence of the uniqueness is the following:

(1.5) Suppose M is a local martingale which is zero at 0 and T is a stopping time. If we can define $W = H_c{\cdot}M$ and $H_c{\cdot}M^T$, then $W^T = H_c{\cdot}M^T$.

To prove this result, we first note that for every bounded real martingale Y,

$$Y(H_c\cdot M^T) - H\cdot[M^T, Y]$$

is a local martingale which is zero at 0, and this property uniquely characterizes $H_c\cdot M^T$.

We assert that for every bounded martingale Y,

$$YW^T - H\cdot[M^T, Y]$$

is also a local martingale, zero at 0.

We do know that $YW - H\cdot[M, Y]$ is a local martingale which is zero at 0; thus, we let (T_n) be a sequence of stopping times reducing both W and $YW - H\cdot[M, Y]$.

We claim that (T_n) also reduces $YW^T - H\cdot[M^T, Y]$.

We have that $B^n = Y^{T_n}W^{T_n} - \int^{t\wedge T_n} H_s d[M, Y]_s$ is a uniformly integrable martingale, zero at 0. Hence, for every stopping time S, $E[B^n_S] = 0$; moreover, $E[B^n_{S\wedge T}] = 0$. That is,

$$(1.6) \qquad E[Y_{S\wedge T\wedge T_n}W_{S\wedge T\wedge T_n} - \int^{S\wedge T\wedge T_n} H_s d[M, Y]_s] = 0.$$

However, since W^{T_n} is also a uniformly integrable martingale, $W_{S\wedge T\wedge T_n}$ is integrable and then so is $Y_{S\wedge T\wedge T_n}W_{S\wedge T\wedge T_n}$, since Y is bounded. Thus, (1.6) becomes

$$(1.7) \qquad E[Y_{S\wedge T\wedge T_n}W_{S\wedge T\wedge T_n}] - E[\int^{S\wedge T\wedge T_n} H_s d[M, Y]_s] = 0.$$

However,

$$\begin{aligned} E[Y_{S\wedge T\wedge T_n} W_{S\wedge T\wedge T_n}] &= E[E(Y_{S\wedge T_n} | F_T) W_{S\wedge T\wedge T_n}] \\ &= E[E(Y_{S\wedge T_n} W_{S\wedge T\wedge T_n} | F_T)] \\ &= E[Y_{S\wedge T_n} W_{S\wedge T\wedge T_n}]. \end{aligned}$$

Thus, for every stopping time S,

$$\begin{aligned} &E[Y_S^{T_n} W_S^{T\wedge T_n} - \int^{S\wedge T_n} H_s d[M^T, Y]_s] \\ &\qquad = E[Y_{S\wedge T_n} W_{S\wedge T\wedge T_n} - \int^{S\wedge T\wedge T_n} H_s d[M, Y]_s] \\ &\qquad = 0. \end{aligned}$$

That is, $Y^{T_n} W^{T\wedge T_n} - (H\cdot[M^T, Y])^{T_n}$ is a uniformly integrable martingale. Hence, T_n reduces $YW^T - H\cdot[M^T, Y]$.

Hence, $(H\underset{c}{\cdot}M)^T = W^T = H\underset{c}{\cdot}M^T$. A second consequence of the uniqueness is the following:

(1.8) Suppose M and N are local martingales, zero at 0, and $H\underset{c}{\cdot}M$ and $H\underset{c}{\cdot}N$ are defined, then $H\underset{c}{\cdot}(M+N) = H\underset{c}{\cdot}M + H\underset{c}{\cdot}N$.

For if Y is real bounded martingale, then

$$\begin{aligned} &Y(H\underset{c}{\cdot}M + H\underset{c}{\cdot}N) - H\cdot[M+N, Y] \\ &\qquad = Y(H\underset{c}{\cdot}M) - H\cdot[M, Y] + Y(H\underset{c}{\cdot}N) - H\cdot[N, Y] \end{aligned}$$

is a local martingale which is zero at 0. We now construct $H \underset{c}{\cdot} M$ for bounded H.

We write $M = N + B$, where N is a local martingale with bounded jumps, B is a local martingale of finite variation, and $N_0 = 0 = B_0$.

We let (T_n) be a sequence of stopping times which increase to infinity such that N^{T_n} is a square integrable martingale for each n. Since H is bounded, H is in $L_0^2(N^{T_n})$ and $H \underset{c}{\cdot} N^{T_n}$ exists, as defined in Theorem 1.2. However, by Theorem 1.4 (c), $Y(H \underset{c}{\cdot} N^{T_n}) - H \cdot [N^{T_n}, Y]$ is a uniformly integrable martingale, zero at 0, for every bounded martingale Y; hence, $H \underset{c}{\cdot} N^{T_n}$ is unambiguously and uniquely defined and if H is predictable, then $H \underset{c}{\cdot} N^{T_n} = H \cdot N^{T_n}$. Moreover, by (1.5)

$$(H \underset{c}{\cdot} N^{T_{n+1}})^{T_n} = H \underset{c}{\cdot} N^{T_n}$$

and we may define $H \underset{c}{\cdot} N = \lim_n H \underset{c}{\cdot} N^{T_n}$, by pasting together, so that $(H \underset{c}{\cdot} N)^{T_n} = H \underset{c}{\cdot} N^{T_n}$. We must show $H \underset{c}{\cdot} N$ satisfies the theorem.

Let Y be a bounded martingale. Then Y^{T_n} is still a bounded martingale; hence,

$$Y^{T_n}(H \underset{c}{\cdot} N^{T_n}) - H \cdot [N^{T_n}, Y^{T_n}]$$

is a uniformly integrable martingale, zero at 0. However,

$$Y^{T_n}(H \underset{c}{\cdot} N^{T_n}) - H\cdot[N^{T_n}, Y^{T_n}] = Y^{T_n}(H \underset{c}{\cdot} N)^{T_n} - H\cdot[N, Y]^{T_n}$$

$$= Y^{T_n}(H \underset{c}{\cdot} N)^{T_n} - (H\cdot[N, Y])^{T_n}.$$

Thus, $Y(H \underset{c}{\cdot} N) - H\cdot[N, Y]$ is a local martingale, reduced by (T_n), and $H \underset{c}{\cdot} N$ satisfies the theorem.

Moreover, if H is predictable, $H \underset{c}{\cdot} N = H\cdot N$ by Theorem II.2.3.

(1.9) Since B is a local martingale, zero at 0, and is also of finite variation, B is locally integrable (and $B^p = 0$). That is, $\int^t |dB_s|$ is locally integrable. As H is bounded, $\int^t |H_s||dB_s|$ is locally integrable. We let $C_t = \int^t H_s dB_s$. By Lemma 2.5, $H \underset{c}{\cdot} B$ is defined and $H \underset{c}{\cdot} B = C - C^p$. Moreover, if H is predictable, $C^p = H\cdot B^p = 0$; hence, $H \underset{c}{\cdot} B = C = H\cdot B$.

Thus, by (1.8), $H \underset{c}{\cdot} M = H \underset{c}{\cdot} N + H \underset{c}{\cdot} B$ is also defined and $H \underset{c}{\cdot} M = H\cdot M$ if H is predictable.

Before we extend to locally bounded H, we consider the following additional consequence of the uniqueness.

(1.10) Suppose M is a local martingale, zero at 0, T is a stopping time, and $H \underset{c}{\cdot} M^T$ and $H1_{[0,T]} \underset{c}{\cdot} M$ are defined. Then $H \underset{c}{\cdot} M^T = H1_{[0,T]} \underset{c}{\cdot} M$.

This result is immediate since for every bounded martingale Y,

$$Y(H \underset{c}{\cdot} M^T) - H1_{[0,T]}\cdot[M, Y] = Y(H \underset{c}{\cdot} M^T) - H\cdot[M, Y]^T$$

$$= Y(H\underset{c}{\cdot}M^T) - H\cdot[M^T, Y]$$

is a local martingale, zero at 0.

Now suppose H is locally bounded. There exists stopping times (T_n) which increase to infinity such that $H1_{]0,T_n]}$ is bounded.

Hence, $W^n = H1_{]0,T_n]}\underset{c}{\cdot}M$ exists, for any local martingale M which is zero at 0. But since $M_0 = 0$, we have for every bounded martingale Y that $[M, Y]_0 = 0$ and hence that $H1_{]0,T_n]}\cdot[M, Y] = H1_{[0,T_n]}\cdot[M, Y]$. Thus, $YW^n - H1_{[0,T_n]}\cdot[M, Y]$ is a local martingale which is zero at 0. In other words

(1.11) $H1_{[0,T_n]}\underset{c}{\cdot}M$ exists and $H1_{[0,T_n]}\underset{c}{\cdot}M = H1_{]0,T_n]}\underset{c}{\cdot}M$, for every n.

Then applying (1.5) and (1.10), we have

$$(H1_{[0,T_{n+1}]}\underset{c}{\cdot}M)^{T_n} = H1_{[0,T_{n+1}]}\underset{c}{\cdot}M^{T_n}$$

$$= H1_{[0,T_n]}\underset{c}{\cdot}M.$$

We may then define $H\underset{c}{\cdot}M = \lim_n H1_{[0,T_n]}\underset{c}{\cdot}M$, by pasting, so that

$$(H\underset{c}{\cdot}M)^{T_n} = H1_{[0,T_n]}\underset{c}{\cdot}M.$$

We conclude by showing that $H\underset{c}{\cdot}M$ satisfies the theorem. For every bounded martingale Y,

$$\begin{aligned}(Y(H\cdot_c M) - H\cdot[M, Y])^{T_n} &= Y^{T_n}(H\cdot_c M)^{T_n} - H1_{[0,T_n]}\cdot[M, Y]^{T_n} \\ &= Y^{T_n}(H1_{[0,T_n]}\cdot_c M) - H1_{[0,T_n]}\cdot[M, Y^{T_n}]\end{aligned}$$

is a local martingale, zero at 0; hence, $Y(H\cdot_c M) - H\cdot[M, Y]$ is also, since (T_n) increase to infinity. Moreover, if H is predictable, then $H1_{[0,T_n]}$ is also predictable; hence, $H1_{[0,T_n]}\cdot_c M = H1_{[0,T_n]}\cdot M$, for each n. Finally, $H\cdot_c M = H\cdot M$ by pasting.

We note that lines (1.5), (1.8), (1.9), and (1.10) in the proof are actually theorems in their own right. We shall state these results explicitly for a semimartingale in the next section.

2 COROLLARY. Let $X = X_0 + M + A$ be a special semimartingale and let H be locally bounded. Then for every f in E, $(f, H\cdot_c X) = (f, H)\cdot_c X$.

Proof: Clearly, $(f, H\cdot A) = (f, H)\cdot A$. Let Y be a real, bounded martingale. Then $Y(H\cdot_c M) - H\cdot[M, Y]$ is a local martingale, zero at 0. But then

$$Y(f, H\cdot_c M) - (f, H)\cdot[M, Y] = (f, Y(H\cdot_c M) - H\cdot[M, Y])$$

is a real local martingale, zero at 0. Hence, $(f, H\cdot_c M) = (f, H)\cdot_c M$ and it immediately follows that $(f, H\cdot_c X) = (f, H)\cdot_c X$.

4. Properties of the Optional Integral

We conclude this chapter with some properties of the optional integral. These properties will be easily proved, in view of

Corollary 3.2 and the details in the proof of Theorem 3.1. Throughout this section, H is a locally bounded, optional, E-valued process and X is a special semimartingale with canonical decomposition $X = X_0 + M + A$.

1 THEOREM.

(a) If Z is another special semimartingale, then

$$H \underset{c}{\bullet} (X+Z) = H \underset{c}{\bullet} X + H \underset{c}{\bullet} Z.$$

(b) If K is another locally bounded, optional, E-valued process, then $(H+K) \underset{c}{\bullet} X = H \underset{c}{\bullet} X + K \underset{c}{\bullet} X$.

Proof:

(a) Write $Z = Z_0 + N + B$, the canonical decomposition of Z. Then $X + Z = (X_0 + Z_0) + (M + N) + (A + B)$. Obviously, $H \bullet (A + B) = H \bullet A + H \bullet B$, and by line (1.8) in the proof of Theorem 3.1, we have $H \underset{c}{\bullet} (M + N) = H \underset{c}{\bullet} M + H \underset{c}{\bullet} N$. Thus, (a) follows.

(b) Again it is clear that $(H + K) \bullet A = H \bullet A + K \bullet A$. Since for every bounded martingale Y,

$$\begin{aligned} &Y(H \underset{c}{\bullet} M + K \underset{c}{\bullet} M) - (H + K) \bullet [M, Y] \\ &\qquad = Y(H \underset{c}{\bullet} M) - H \bullet [M, Y] + Y(K \underset{c}{\bullet} M) - K \bullet [M, Y] \end{aligned}$$

is a local martingale, zero at 0, we have $H \underset{c}{\bullet} M + K \underset{c}{\bullet} M = (H + K) \underset{c}{\bullet} M$, and (b) follows.

2 THEOREM. For every stopping time T, $(H\underset{c}{\cdot}X)^T = H\underset{c}{\cdot}X^T = H1_{[0,T]}\underset{c}{\cdot}X$.

Proof: This result follows immediately from lines (1.5) and (1.10) of Theorem 3.1 and from the fact that $(H\cdot A)^T = H\cdot A^T = H1_{[0,T]}\cdot A$.

3 THEOREM. $(H\underset{c}{\cdot}X)_0 = H_0X_0$ and $\Delta(H\underset{c}{\cdot}X) = H\Delta M - {}^p(H\Delta M) + H\Delta A$ on $]0, \infty[$.

Proof: The first assertion is obvious. Since $\Delta(H\cdot A) = H\Delta A$, we need only verify that $\Delta(H\underset{c}{\cdot}M) = H\Delta M - {}^p(H\Delta M)$ on $]0, \infty[$. The result is known for real-valued processes; thus, we may apply Corollary 3.2, provided ${}^p(H\Delta M)$ exists.

Since H is locally bounded, the process $(\int^t |H_s|^2 d[M, M]_s)^{1/2}$ is locally integrable (see the analogous argument in remark (b) after Definition III.3.2); hence, as in Theorem III.4.3, ${}^p(H\Delta M)$ exists.

Then for every f in E, we have on $]0, \infty[$

$$\begin{aligned}(f, H\Delta M - {}^p(H\Delta M)) &= (f, H)\Delta M - {}^p((f, H)\Delta M)\\ &= \Delta(f, H)\underset{c}{\cdot}M\\ &= (f, \Delta(H\underset{c}{\cdot}M)).\end{aligned}$$

Hence, $H\Delta M - {}^p(H\Delta M)$ and $\Delta(H\underset{c}{\cdot}M)$ remain indistinguishable on $]0, \infty[$ in the separable space E.

4 THEOREM.

(a) If K is a real, predictable, locally bounded process, then $H\underset{c}{\cdot}(K\cdot X) = HK\underset{c}{\cdot}X = K\cdot(H\underset{c}{\cdot}X)$.

(b) If H is a real, locally bounded, optional process and K is an E-valued, locally bounded, predictable process, then $H \underset{c}{\cdot} (K \cdot X) = HK \underset{c}{\cdot} X = K \cdot (H \underset{c}{\cdot} X)$.

Proof:

(a) The proof is identical to Theorem 1.7.

(b) Using Corollaries III.3.4 and 3.2, for every f in E we have

$$\begin{aligned} (f,\ H \underset{c}{\cdot} (K \cdot X)) &= H \underset{c}{\cdot} (f,\ K \cdot X) \\ &= H \underset{c}{\cdot} ((f,\ K) \cdot X) \\ &= H(f,\ K) \underset{c}{\cdot} X \\ &= (f,\ HK) \underset{c}{\cdot} X \\ &= (f,\ HK \underset{c}{\cdot} X) \end{aligned}$$

and

$$\begin{aligned} (f,\ H \underset{c}{\cdot} (K \cdot X)) &= H(f,\ K) \underset{c}{\cdot} X \\ &= (f,\ K) \cdot (H \underset{c}{\cdot} K) \\ &= (f,\ K \cdot (H \underset{c}{\cdot} X)). \end{aligned}$$

Hence, $H \underset{c}{\cdot} (K \cdot X)$, $HK \underset{c}{\cdot} X$, and $K \cdot (H \underset{c}{\cdot} X)$ are all indistinguishable in E.

We remark that as in Theorem III.4.6, if X is quasi-left-continuous, then Theorem 3 becomes $\Delta(H \underset{c}{\cdot} X) = H\Delta X$. If $X = X_0 + M$ is quasi-left-continuous, then Theorem 4 holds for optional K, as in Corollary III.4.5.

5 THEOREM. Suppose M is a real local martingale, zero at 0, that is also a process of finite variation. Then for every locally bounded, optional, E-valued process H, $\int^t |H_s||dM_s|$ is locally integrable and $H_{\dot{c}}M = C - C^p$, where $C_t = \int^t H_s dM_s$.

Proof: The process $\int^t |dM_s|$ is locally integrable, since any local martingale of finite variation is locally integrable. Since H is locally bounded, $\int^t |H_s||dM_s|$ remains locally integrable. The theorem then follows from Lemma 2.5.

6 COROLLARY. Let M and H be as in Theorem 5. Then $\|H_{\dot{c}}M\|_{H^1} \leq 2E[\int^\infty |H_s||dM_s|]$.

Proof. Since $H_{\dot{c}}M$ is a local martingale that is also of finite variation, by Theorem II.5.2,

$$\begin{aligned}\|H_{\dot{c}}M\|_{H^1} &\leq E[\int^\infty |d(H_{\dot{c}}M)_s|) \\ &\leq E[\int^\infty |dC_s|] + E[\int^\infty |dC_s^p|]\end{aligned}$$

where $C = \int^t H_s dM_s$, as in Theorem 5. Thus, it suffices to show $E[\int^\infty |dC_s^p|] \leq E[\int^\infty |dC_s|]$, since $\int^\infty |dC_s| = \int^\infty |H_s||dM_s|$.

Since stopping and compensation commute, we may localize and assume that C is of integrable variation. Consider then the E-valued P-measure m, determined by C, defined on the predictable σ-algebra

$$m(K) = E[\int^\infty K_s dC_s],$$

for every real, bounded, predictable K. Then $|m|(K) = E[\int^\infty K_s |dC_s|]$.

Let Q_s be a predictable density given by

$$Q_s = \frac{dm}{d|m|} .$$

Then, $|Q_s| = 1$, $|m|$-almost everywhere.

Let D be the predictable dual projection of $\int^t |dC_s|$. Then for every bounded, measurable K,

$$E[\int^{\infty p} K_s dC_s] = E[\int^{\infty p} K_s Q_s |dC_s|]$$

$$= E[\int^{\infty p} (K_s Q_s) |dC_s|]$$

$$= E[\int^{\infty} K_s Q_s dD_s].$$

Thus, $C^p = \int^t Q_s dD_s$. Moreover, since D is increasing

$$E[\int^{\infty} |dC_s^p|] = E[\int^{\infty} |Q_s| dD_s]$$

$$= E[\int^{\infty} |Q_s| |dC_s|]$$

$$= E[\int^{\infty} |dC_s|],$$

and the result is proved.

We note that the proof actually contains a separate theorem. Namely, $E[\int^{\infty} |dC_s^p|] = E[\int^{\infty} |dC_s|]$, for any locally integrable process C.

We conclude this section and chapter with a convergence theorem and the decomposition theorem for the optional integral.

6 THEOREM. Let (H^k) be a sequence of optional E-valued processes converging pointwise to H, and assume that (H^k) are uniformly bounded in norm by a locally bounded process K. Then $(H^k \underset{c}{\cdot} X)$ converges locally in S^1 to $H \underset{c}{\cdot} X$.

Proof: We have $X = X_0 + M + A$, the canonical decomposition. We decompose $M = N + B$, where N is a locally square integrable local martingale, B is a local martingale of finite variation, and $N_0 = 0 = B_0$.

We choose stopping times (T_n) which increase to infinity such that for each n

(i) N^{T_n} is a square integrable martingale,

(ii) $K1_{]0,T_n]}$ is bounded,

(iii) $\int^{T_n} |K_s||dB_s|$ is integrable,

(iv) $\int^{T_n} |K_s||dA_s|$ is integrable, and

(v) $K_0 X_0 1_{(T_n>0)}$ is integrable.

B is locally integrable, being a local martingale of finite variation, and A is locally integrable, being a predictable process of finite variation; hence, $\int^t |K_s||dB_s|$ and $\int^t |K_s||dA_s|$ remain locally integrable, since K is locally bounded. In other words, conditions (iii) and (iv) may be satisfied. As for condition (v), $Y = K_0(X_0 + M)$ is still a local martingale; so choosing T_n which reduces Y, we have $Y_0 1_{(T_n>0)}$ is integrable. Hence, condition (v) may be satisfied.

We now fix n and denote $T = T_n$.

By the same argument as in line (1.11) of the proof of Theorem 3.1, we have

$$(H \overset{\bullet}{c} N)^T = H1_{[0,T]} \overset{\bullet}{c} N^T = H1_{]0,T]} \overset{\bullet}{c} N^T,$$

with the first equality coming from Theorem 2.

But $H1_{]0,T]}$ is bounded by $K1_{]0,T]}$ which is simply bounded. Thus, $H1_{]0,T]}$ is in $L_0^2(N^T)$.

Let $W = (H \overset{\bullet}{c} N)^T = H1_{]0,T]} \overset{\bullet}{c} N^T$. Then

$$\begin{aligned} (6.1) \qquad \|W\|_{H^1} &\le \|W\|_{H^2} \\ &= \|W\|_{M^2} \\ &= (E[[W, W]_\infty])^{1/2} \\ &\le (E[\int^\infty |H_s|^2 1_{]0,T]} d[N^T, N^T]_s])^{1/2} \\ &< \infty. \end{aligned}$$

The expression is finite since $H1_{]0,T]}$ is bounded and N^T is a square integrable martingale. The preceding inequality is Theorem 1.4 (b).

By Corollary 5, we have

$$(6.2)\qquad \|(H\underset{c}{\cdot}B)^T\|_{H^1} = \|H\underset{c}{\cdot}B^T\|_{H^1}$$

$$\le 2E[\int^\infty |H_s||dB^T_s|]$$

$$= 2E[\int^T |H_s||dB_s|]$$

$$\le 2E[\int^T |K_s||dB_s|]$$

$$< \infty.$$

Since,

$$(H\underset{c}{\cdot}X)^T 1_{(T>0)} = H_0X_01_{(T>0)} + (H\underset{c}{\cdot}M)^T + (H\cdot A)^T$$

$$= H_0X_01_{(T>0)} + (H\underset{c}{\cdot}N)^T + (H\underset{c}{\cdot}B)^T + H\cdot A^T,$$

we have

$$\|(H\cdot X)^T 1_{(T>0)}\|_{S^1}$$

$$\le \|H_0X_01_{(T>0)}\|_{L^1} + \|(H\underset{c}{\cdot}N)^T + (H\underset{c}{\cdot}B)^T\|_{H^1} + E[\int^\infty |H_s||dA^T_s|]$$

$$\le \|K_0X_01_{(T>0)}\|_{L^1} + \|(H\underset{c}{\cdot}N)^T\|_{H^1} + \|(H\underset{c}{\cdot}B)^T\|_{H^1}$$

$$+ E[\int^T |K_s||dA_s|]$$

$$< \infty.$$

Thus, $(H \cdot_c X)^T 1_{(T>0)}$ is in S^1. Likewise, $(H^k \cdot_c X)^T 1_{(T>0)}$ is in S^1 for each k.

Then letting $L^k = H - H^k$, we have

$$\|(H \cdot_c X)^T 1_{(T>0)} - (H^k \cdot_c X)^T 1_{(T>0)}\|_{S^1}$$

$$= \|(L^k \cdot_c X)^T 1_{(T>0)}\|_{S^1}$$

$$\leq \|L_0^k X_0 1_{(T>0)}\|_{L^1} + \|(L^k \cdot_c N)^T\|_{H^1} + \|(L^k \cdot_c B)^T\|_{H^1}$$

$$+ E[\int^T |L_s^k||dA_s|]$$

$$\leq \|L_0^k X_0 1_{(T>0)}\|_{L^1} + (E[\int^\infty |L_s^k|^2 1_{]0,T]} d[N^T, N^T]_s])^{1/2}$$

$$+ 2E[\int^T |L_s^k||dB_s|] + 2E[\int^T |L_s^k||dA_s|].$$

Letting k tend to infinity, we have by dominated convergence that $(H^k \cdot_c X)^T 1_{(T>0)}$ converges to $(H \cdot_c X)^T 1_{(T>0)}$ in S^1, and the theorem is proved.

7 THEOREM. Let X be a local martingale and let H be locally bounded. Then $(H \cdot_c X)^c = H \cdot_c X^c$ and $(H \cdot_c X)^d = H \cdot_c X^d$. The same result holds if X is in M^2 and H is in $L_0^2(X)$.

Proof: Once again we apply Corollary 3.2. However, we first note that Theorem III.5.10 proves the result for H real and X E-valued and in particular for H and X both real. Also applying Corollary III.5.6, we have for every f in E,

$$\begin{aligned}(f,\ H \underset{c}{\cdot} X^{c}) &= (f,\ H) \underset{c}{\cdot} X^{c}\\ &= ((f,\ H) \underset{c}{\cdot} X)^{c}\\ &= ((f,\ H \underset{c}{\cdot} X))^{c}\\ &= (f,\ (H \underset{c}{\cdot} X)^{c}).\end{aligned}$$

Hence, $H \underset{c}{\cdot} X^{c}$ and $(H \underset{c}{\cdot} X)^{c}$ are indistinguishable in E. Similarly, $H \underset{c}{\cdot} X^{d} = (H \underset{c}{\cdot} X)^{d}$.

CHAPTER V
THE OPTIONAL STOCHASTIC INTEGRAL IN HILBERT SPACE

In this chapter, we develop the stochastic integral of an optional Hilbert-valued process with respect to a Hilbert-valued special semimartingale. This integral will be defined for all locally bounded processes and will again be an extension of the predictable stochastic integral. We shall begin by defining the integral of a bounded process with respect to a square integrable martingale. The construction in this case may also be applied to the optional integrals defined in Sections III.1 and IV.1 and, in particular, to the classical optional integral for real processes defined in Dellacherie and Meyer [4]. Besides offering a unified approach in all settings for locally bounded processes, this construction more importantly gives a more natural definition of the optional integral. Specifically, we approximate, in some sense, an optional process H by a predictable process H^1 and obtain $H \underset{c}{\cdot} X = H^1 \cdot X + L$, where L is a local martingale obtained by properly "integrating" $H - H^1$. Unless otherwise specified, H and X will be E-valued throughout the chapter.

1. The Optional Integral With Respect to a Square Integrable Martingale

We define in this section the optional integral of an E-valued optional process H with respect to an E-valued square integrable martingale X. The approach will be slightly different than for the integrals defined in Sections III.1 and IV.1. In these previous sections, we defined the integral in terms of a characterizing property which also established its uniqueness. The integral here will first be defined as a sum of a predictable stochastic integral and a square integrable martingale. Its uniqueness will be established by showing the independence of the choice of such a representation. We shall then give a characterizing property which alternately establishes uniqueness.

1 LEMMA. Let H be a bounded, optional process. There exists a predictable process H^1 such that $H_0 = H_0^1$ a.s. and $H - H^1$ is thin. Moreover, H^1 may be taken to have the same bound as H.

Proof: Since H is bounded, ${}^pH$ exists [see 2]. Then for every predictable time T,

$$ {}^pH_T 1_{(T<\infty)} = E(H_T 1_{(T<\infty)} | F_{T-}) \text{ a.s.} $$

Taking T = 0 and recalling that $F_{0-} = F_0$, we have ${}^pH_0 = H_0$ a.s. For every f in E we have $(f, {}^pH) = {}^p(f, H)$. Then by Déllacherie and Meyer [4, VI.46], the set $\{(f, H) - {}^p(f, H) \neq 0\}$ is a countable union of graphs of stopping times. Then the set $\{H - {}^pH \neq 0\}$ is contained in a countable union of graphs of stopping times, for if we let (f_n) be a countable norming set of E, then

$$\{H - {}^pH \neq 0\} \subseteq \bigcup_n \{(f_n, H - {}^pH) \neq 0\}$$

$$= \bigcup_n \{(f_n, H) - {}^p(f_n, H) \neq 0\}.$$

By Déllacherie and Meyer [3, IV.87 (a)], $\{H - {}^pH \neq 0\}$ is the countable union of disjoint graphs of stopping times, and $H - {}^pH$ is thin.

Moreover, if $|H| \leq k$, then for every predictable time T,

$$E[|{}^pH_T| 1_{(T<\infty)}] \leq k$$

and ${}^pH$ is bounded by k by the cross-section theorem.

(a) We point out that the result remains valid for H taking values in any separable Banach space.

(b) We also note that if H is bounded by k and L is any predictable process such that H - L is thin, then we may let $L^1 = H_0 1_{[0]} + L 1_{\{|L| \leq k\}} 1_{]0,\infty[}$. Then L^1 is bounded by k, L^1 is still predictable, and $H - L^1$ is still thin.

2 LEMMA. Let X be a semimartingale and let K be an E-valued, predictable, thin, bounded process such that $\sum_{s \leq t} |(K_s, \Delta X_s)| < \infty$ for every t. Then $K \cdot X = \sum_{s \leq t} (K_s, \Delta X_s)$.

Proof: The set $\{K \neq 0\}$ is the countable union of disjoint graphs, $[T_n]$, of a sequence of predictable stopping times (T_n).

Let $K^j = \sum_{n\leq j} K_{T_n} 1_{[T_n]}$. Then for every t,

$$(K^j - X)_t = \sum_{n\leq j} (K_{T_n} 1_{[T_n]} \cdot X)_t$$

$$= \sum_{n\leq j} (K_{T_n}, X_t^{T_n} - X_t^{T_n-})$$

$$= \sum_{n\leq j} (K_{T_n}, \Delta X_{T_n}) 1_{(T_n\leq t)}.$$

Hence,

$$\lim_j (K^j \cdot X)_t = \sum_{n=1}^{\infty} (K_{T_n}, \Delta X_{T_n}) 1_{(T_n\leq t)}$$

$$= \sum_{s\leq t} (K_s, \Delta X_s).$$

On the other hand, (K^j) are uniformly bounded and

$$\lim_j K^j = \sum_{n=1}^{\infty} K_{T_n} 1_{[T_n]} = K;$$

hence, for every t, $(K^j \cdot X)_t$ converges in probability to $(K \cdot X)_t$, as j tends to infinity (Theorem II.2.8). Hence, $(K \cdot X)_t = \sum_{s\leq t} (K_s, \Delta X_s)$ a.s. for each t. By right continuity, the processes are indistinguishable.

We remark that we may take one or both of K and X to be real-valued and Lemma 2 remains valid.

3 THEOREM. Let B be an E-valued process of integrable variation such that $E[[B, B]_\infty] < \infty$ and $B - B^p$ is a square integrable martingale. Then for every E-valued square integrable martingale Y, $E[(Y_\infty, B_\infty - B^p_\infty)] = E[\int_{]0,\infty]} \Delta Y_s dB_s]$.

Proof: Since Y and $B - B^p$ are in M^2_E, $E[(Y_\infty, B_\infty - B^p_\infty)] = E[[Y, B-B^p]_\infty]$ exists and is finite.

By the Kunita-Watanabe inequality followed by Holder's inequality, we have

$$E[\int^\infty |d[Y, B]_s|] \leq (E[[Y, Y]_\infty])^{1/2}(E[[B, B]_\infty])^{1/2}$$
$$\leq \infty.$$

Thus, $[Y, B]_\infty$ is integrable. Since, $[Y, B-B^p]_\infty$ is also integrable, we have that $[Y, B^p]_\infty$ is integrable. Then

$$(3.1) \qquad E[[Y, B]_\infty] = E[\sum_{s<\infty} (\Delta Y_s, \Delta B_s)]$$
$$= E[\int^\infty \Delta Y_s dB_s].$$

Since $E[[Y, B-B^p]_\infty] = E[[Y, B]_\infty] - E[[Y, B^p]_\infty]$, it suffices to show that $E[[Y, B^p]_\infty] = E[(Y_0, B_0)]$. By Theorem II.4.4, we have $E[[B^p, B^p]_\infty] \leq E[[B, B]_\infty] < \infty$; thus, as in (3.1),

$$(3.2) \qquad E[\int^\infty |d[Y, B^p]_s|] < \infty.$$

By Theorem II.3.4, $[Y, B^p]$ is a local martingale. However, for every stopping time T,

$$
\begin{aligned}
|[Y, B^p]_T 1_{(T<\infty)}| &\leq \int^T |d[Y, B^p]_s| \\
&\leq \int^\infty |d[Y, B^p]_s|;
\end{aligned}
$$

hence, the set $\{[Y, B^p]_T 1_{(T<\infty)} : T \text{ is a stopping time}\}$ is uniformly integrable. $[Y, B^p]$ is then a local martingale of class D and is, therefore, a true uniformly integrable martingale. Thus,

$$
\begin{aligned}
E[[Y, B^p]_\infty] &= E[[Y, B^p]_0] \\
&= E[(Y_0, B_0^p)] \\
&= E[(Y_0, B_0)],
\end{aligned}
$$

and

$$
\begin{aligned}
E[(Y_\infty, B_\infty - B_\infty^p)] &= E[[Y, B-B^p]_\infty] \\
&= E[\int^\infty \Delta Y_s dB_s] - E[(Y_0, B_0)] \\
&= E[\int_{]0,\infty]} \Delta Y_s dB_s].
\end{aligned}
$$

4 THEOREM. Let X be a square integrable martingale, let H be bounded and optional, and let H^1 be bounded and predictable such that $H_0^1 = H_0$ and $H - H^1$ is thin. For each n, we set

$$B_t^n = \sum_{s \leq t} (H_s - H_s^1, \Delta X_s) 1_{\{|\Delta X_s| > \frac{1}{n}\}}.$$

Then for each n

(a) B^n is locally integrable, and

(b) $B^n - (B^n)^p$ is a square integrable martingale. Moreover, the sequence $(B^n - (B^n)^p)$ is Cauchy in M^2.

Proof: We first note that since X is cadlag, it has only a finite number of jumps of magnitude greater than 1/n on each finite interval [0, t]. Hence, B^n_t is a well-defined process of finite variation.

Moreover, since $H^1_0 = H_0$, we have

$$B^n_t = \sum_{0<s\le t} (H_s - H^1_s, \Delta X_s) 1_{\{|\Delta X_s| > \frac{1}{n}\}}.$$

We may then assume $X_0 = 0$. For if we replace X by $X - X_0$, B^n is unchanged since $\Delta X = \Delta(X - X_0)$ on $]0, \infty]$.

We may further assume that H and H^1 are bounded by the same constant k. Set

$$C^n_t = \sum_{s\le t} |\Delta X_s| 1_{\{|\Delta X_s| > \frac{1}{n}\}}.$$

Then

$$\sum_{s\le t} |(H_s - H^1_s, \Delta X_s)| 1_{\{|\Delta X_s| > \frac{1}{n}\}} \le 2kC^n_t;$$

thus, it suffices to show C^n_t is locally integrable. However,

$$C^n_t = C^n_{t-} + |\Delta X_t| 1_{\{|\Delta X_t| > \frac{1}{n}\}}$$

$$\le C^n_{t-} + (\Delta[X, X]_t)^{1/2}$$

$$\leq C^n_{t-} + [X, X]^{1/2}_t .$$

Thus, C^n_t is locally integrable, since $[X, X]^{1/2}_t$ is locally integrable by Theorem II.4.5 and C^n_{t-} is locally integrable, since it is increasing and left continuous. Hence, B^n_t is locally integrable for each n.

By Corollary II.4.2 and Theorem II.3.2, we have

$$[B^n-(B^n)^p, B^n-(B^n)^p]_\infty \leq 2[[B^n, B^n]_\infty + [-(B^n)^p, -(B^n)^p]_\infty]$$

$$= 2[[B^n, B^n]_\infty + [(B^n)^p, (B^n)^p]_\infty].$$

Then, by Theorem II.4.4,

$$E[[B^n-(B^n)^p, B^n-(B^n)^p]_\infty] \leq 4E[[B^n, B^n]_\infty]$$

$$\begin{aligned} &= 4E[\sum_{s<\infty} |\Delta B^n_s|^2] &(4.1)\\ &= 4E[\sum_{s<\infty} |(H_s-H^1_s, \Delta X_s)|^2 1_{\{|\Delta X_s| > \frac{1}{n}\}}] \\ &\leq 16k^2E[\sum_{s<\infty} |\Delta X_s|^2] \\ &= 16k^2E[\sum_{s<\infty} \Delta[X, X]_s] \\ &\leq 16k^2E[[X, X]_\infty] \\ &< \infty. \end{aligned}$$

Thus, by Theorem II.3.7, $B^n - (B^n)^p$ is a square integrable martingale for each n. We next note that $\sum_{s<\infty} \Delta[X, X]_s$ is in L^1, since

$$E[\sum_{s<\infty} \Delta[X, X]_s] \leq E[[X, X]_\infty] < \infty.$$

Thus, $\sum_{s<\infty} \Delta[X, X]_s < \infty$ for a.e. w. Hence, the sum $\sum_{s<\infty} \Delta[X, X]_s 1_{\{|\Delta X_s| > \frac{1}{n}\}}$ converges uniformly in n for a.e. w.

By the Moore Lemma, we have a.s.

$$\begin{aligned} \lim_n \sum_{s<\infty} \Delta[X, X]_s 1_{\{|\Delta X_s| > \frac{1}{n}\}} &= \sum_{s<\infty} \lim_n \Delta[X, X]_s 1_{\{|\Delta X_s| > \frac{1}{n}\}} \\ &= \sum_{s<\infty} \Delta[X, X]_s 1_{\{|\Delta X_s| > 0\}} \\ &= \sum_{s<\infty} \Delta[X, X]_s. \end{aligned}$$

By monotone convergence, the limit holds in L^1.

We set $Y^n = B^n - (B^n)^p$. We have, for $n > m$, that $Y^n - Y^m = (B^n - B^m) - (B^n - B^m)^p$; hence, as in (4.1),

$$\begin{aligned} E[[Y^n-Y^m, Y^n-Y^m]_\infty] &\leq 4E[[B^n-B^m, B^n-B^m]_\infty] \\ &= 4E[\sum_{s<\infty} |\Delta(B^n_s - B^m_s)|^2] \\ &= 4E[\sum_{s<\infty} |(H_s-H^1_s, \Delta X_s)|^2 1_{\{\frac{1}{m} \geq |\Delta X_s| > \frac{1}{n}\}}] \\ &\leq 16k^2 E[\sum_{s<\infty} \Delta[X, X]_s 1_{\{\frac{1}{m} \geq |\Delta X_s| > \frac{1}{n}\}}] \end{aligned}$$

$$= 16k^2 E[\sum_{s<\infty} \Delta[X, X]_s 1_{\{|\Delta X_s| > \frac{1}{n}\}} - \sum_{s<\infty} \Delta[X, X]_s 1_{\{|\Delta X_s| > \frac{1}{m}\}}].$$

Hence, (Y^n) is Cauchy in M^2, since

$$\|Y^n - Y^m\|_{M^2} = (E[[Y^n-Y^m, Y^n-Y^m]_\infty])^{1/2}$$

and

$$\lim_{m,n}[E[\sum_{s<\infty} \Delta[X, X]_s 1_{\{|\Delta X_s| > \frac{1}{n}\}} - \sum_{s<\infty} \Delta[X, X]_s 1_{\{|\Delta X_s| > \frac{1}{m}\}}] = 0.$$

Once again, we point out that the preceding theorem remains true if we take one or both of H and X to be real-valued. One simply replaces inner products by products in the proof.

We now come to the definition of the optional stochastic integral.

5 DEFINITION. Let H be a bounded optional process and let X be a square integrable martingale. Let H^1 be any bounded predictable process such that $H^1_0 = H_0$ and $H - H^1$ is thin. We set $B^n_t = \sum_{s\leq t} (H_s-H^1_s, \Delta X_s)1_{\{|\Delta X_s| > \frac{1}{n}\}}$. We define the optional (or compensated) stochastic integral of H with respect to X to be the process $H \underset{c}{\cdot} X = H^1 \cdot X + Z$, where $Z = M^2 - \lim_n (B^n - (B^n)^p)$.

We note that Z exists and is a square integrable martingale by Theorem 4. Since H^1 is bounded, H^1 is in $L_p^2(X)$ and $H^1 \cdot X$ is also a square integrable martingale.

6 THEOREM. $H \underset{c}{\cdot} X$ is a well-defined square integrable martingale, independent of the choice of H^1 in its definition. Moreover, if H is predictable, then $H \underset{c}{\cdot} X = H \cdot X$.

Proof: The last assertion is obvious for if $H \underset{c}{\cdot} X$ is well-defined and H is predictable, then we may take $H^1 = H$ and obtain $H \underset{c}{\cdot} X = H^1 \cdot X = H \cdot X$. Let

$$A_t^n = \sum_{s \leq t} \Delta X_s 1_{\{|\Delta X_s| > \frac{1}{n}\}}.$$

Then

$$C_t^n = \int^t |dA_s^n| = \sum_{s \leq t} |\Delta X_s| 1_{\{|\Delta X_s| > \frac{1}{n}\}}.$$

As in the proof of Theorem 4, we have

$$\begin{aligned} C_t^n - C_0^n &\leq C_{t-}^n + [X, X]_t^{1/2} - |X_0| 1_{\{|\Delta X_0| > \frac{1}{n}\}} \\ &= C_{t-}^n + [X, X]_t^{1/2} - |X_0| + |X_0| 1_{\{|\Delta X_0| \leq \frac{1}{n}\}} \\ &= C_{t-}^n + [X, X]_t^{1/2} - [X, X]_0^{1/2} + |X_0| 1_{\{|\Delta X_0| \leq \frac{1}{n}\}}. \end{aligned}$$

By Theorem II.4.5, $[X, X]_t^{1/2} - [X, X]_0^{1/2}$ is locally integrable (here we are not assuming $X_0 = 0$). C^n_{t-} is still locally integrable, as in Theorem 4, and $|X_0|$ is integrable since X is a martingale. Thus, $C^n_t - C_0$ is locally integrable which means that A^n is locally integrable.

Let

$$U^n = A^n - (A^n)^p + X_0 1_{\{|\Delta X_0| > \frac{1}{n}\}}. \tag{6.1}$$

We claim that $U^n = 1_{\{|\Delta X| > \frac{1}{n}\}} \overset{\cdot}{c} X$. Since

$$\begin{aligned} E[[A^n-(A^n)^p, A^n-(A^n)^p]_\infty] &\le 4E[[A^n, A^n]_\infty] \\ &= 4E[\sum_{s<\infty} |\Delta A^n_s|^2] \\ &= 4E[\sum_{s<\infty} |\Delta X_s|^2 1_{\{|\Delta X_s| > \frac{1}{n}\}}] \\ &\le 4E[\sum_{s<\infty} \Delta[X, X]_s] \\ &\le 4E[[X, X]_\infty] \\ &< \infty, \end{aligned}$$

We have $A^n - (A^n)^p$, and hence U^n, is a square integrable martingale. Applying Theorem 3, we have for every Y in M^2_E,

$$\begin{aligned} &E[(Y_\infty, U^n_\infty)] \\ &\quad = E[(Y_\infty, X_0 1_{\{|\Delta X_0| > \frac{1}{n}\}})] + E[(Y_\infty, A^n_\infty-(A^n)^p_\infty)] \end{aligned}$$

$$= E[E((Y_\infty, X_0 1_{\{|\Delta X_0| > \frac{1}{n}\}})|F_0)] + E[\int_{]0,\infty]} \Delta Y_s dA_s^n]$$

$$= E[(Y_0, X_0 1_{\{|\Delta X_0| > \frac{1}{n}\}})] + E[\sum_{0<s<\infty} (\Delta Y_s, \Delta A_s^n)]$$

$$= E[\sum_{s<\infty} (\Delta Y_s, \Delta X_s) 1_{\{|\Delta X_s| > \frac{1}{n}\}}].$$

On the other hand, ΔX is thin, thus so is $1_{\{|\Delta X| > \frac{1}{n}\}}$; hence,

$$E[\int^\infty 1_{\{|\Delta X_s| > \frac{1}{n}\}} d[X, Y]_s] = E[\sum_{s<\infty} 1_{\{|\Delta X_s| > \frac{1}{n}\}} \Delta[X, Y]_s]$$

$$= E[\sum_{s<\infty} (\Delta Y_s, \Delta X_s) 1_{\{|\Delta X_s| > \frac{1}{n}\}}].$$

Hence, $E[(Y_\infty, U_\infty^n)] = E[\int^\infty 1_{\{|\Delta X_s| > \frac{1}{n}\}} d[X, Y]_s]$ for every Y in M_E^2

and $U^n = 1_{\{|\Delta X| > \frac{1}{n}\}} \overset{\bullet}{c} X$, by Theorem III.1.2. By Theorem III.1.6,

U^n converges to $1_{\{|\Delta X|>0\}} \overset{\bullet}{c} X = X^d$ in M_E^2.

Now suppose H^1 and K^1 are bounded, predictable, with $H_0^1 = H_0 = K_0^1$, such that $H - H^1$ and $H - K^1$ are thin. We may also assume that H, H^1, and K^1 are all bounded by the same constant k. Note then that $K^1 - H^1$ is bounded, thin, and predictable.

Let

$$B_t^n = \sum_{s\le t} (H_s - H_s^1, \Delta X_s) 1_{\{|\Delta X_s| > \frac{1}{n}\}}$$

and

$$C^n_t = \sum_{s\le t} (H_s - K^1_s, \Delta X_s) 1_{\{|\Delta X_s| > \frac{1}{n}\}}.$$

Then $D^n_t = B^n_t - C^n_t = \sum_{s\le t} (K^1_s - H^1_s, \Delta X_s) 1_{\{|\Delta X_s| > \frac{1}{n}\}}$. We must show that

$$H^1\cdot X + M^2 - \lim_n (B^n - (B^n)^p) = K^1\cdot X + M^2 - \lim_n (C^n - (C^n)^p),$$

or equivalently that

$$(K^1 - H^1)\cdot X = M^2 - \lim_n (D^n - (D^n)^p).$$

Consider $(K^1 - H^1)\cdot U^n$, where U^n is defined above in line (6.1). We have for each t

$$\begin{aligned}\sum_{s\le t} |(K^1_s - H^1_s, \Delta U^n_s)| &\le 2k \sum_{s\le t} |\Delta U^n_s| \\ &\le 2k\int^t |dU^n_s| \\ &< \infty,\end{aligned}$$

since U^n is a process of finite variation. Then, by Lemma 2 and the fact that $K^1_0 - H^1_0 = 0$, we have

$$\begin{aligned}(K^1 - H^1)\cdot U^n &= \sum_{s\le t} (K^1_s - H^1_s, \Delta U^n_s) \\ &= \sum_{0<s\le t} (K^1_s - H^1_s, \Delta U^n_s) \\ &= \int_{]0,t]} (K^1_s - H^1_s) dU^n_s\end{aligned}$$

$$= \int_{]0,t]} (K^1_s - H^1_s)dA^n_s - \int_{]0,t]} (K^1_s - H^1_s)d(A^n)^p_s$$

$$= \int_{]0,t]} (K^1_s - H^1_s)dA^n_s - (\int_{]0,\cdot]} (K^1_s - H^1_s)dA^n_s)^p_t$$

$$= \sum_{s\le t} (K^1_s-H^1_s, \Delta A^n_s) - (\sum_{s\le\cdot} (K^1_s-H^1_s, \Delta A^n_s))^p_t$$

$$= D^n - (D^n)^p.$$

Since U^n converges to X^d in M^2_E, we have, by Theorems II.2.1 and II.2.2 (a), that

$$\|(K^1 - H^1)\cdot U^n - (K^1 - H^1)\cdot X^d\|_{M^2}$$

$$\le (E[\int^\infty (K^1_s - H^1_s)^2 d[U^n-X^d, U^n-X^d]_s])^{1/2}$$

$$\le 2k\|U^n - X^d\|_{M^2_E}.$$

Hence, taking the limit as n tends to infinity in (6.2), we have

(6.3) $$(K^1 - H^1)\cdot X^d = M^2 - \lim_n D^n - (D^n)^p.$$

Finally, we have that for each t

$$\sum_{s\le t} |(K^1_s-H^1_s, \Delta X^c_s)| = 0;$$

hence, by Lemma 2,

$$(K^1 - H^1)\cdot X^c = \sum_{s\le t} |(K_s^1 - H_s^1, \Delta X_s^c)| = 0.$$

Adding $(K^1 - H^1)\cdot X^c$ to the left hand side in (6.3), we obtain

$$(K^1 - H^1)\cdot X = M^2 - \lim_n (D^n - (D^n)^p)$$

and the theorem is proved.

7 REMARKS. We note that Definition 5 and Theorem 6 apply when one or both of H and X are real valued; thus, we have an alternate, and now unified, method for constructing the previous optional integrals in the case of bounded H. Of course, we must show that Definition 5 yields the same process as the previous definitions.

(a) Let H be real, bounded, and optional and let X be E-valued (in particular, X may be real). Define $H\underset{c}{\cdot}X$ as in Definition 5. Then taking H and H^1 to be bounded by k, we have for every Y in M_E^2,

$$\sum_{s<\infty} |H_s - H_s^1||(\Delta X_s, \Delta Y_s)| \le 2k \sum_{s<\infty} |\Delta[X, Y]_s|$$

$$\le 2k\int^\infty |d[X, Y]_s|.$$

The last term is integrable, since X and Y are in M_E^2, hence finite almost surely. Thus, the sum

$$W_\infty^n = \sum_{s<\infty} (H_s - H_s^1)(\Delta X_s, \Delta Y_s)1_{\{|\Delta X_s| > \frac{1}{n}\}}$$

is absolutely convergent uniformly in n, for a.e. w. So by the Moore Lemma, we have a.s.

$$\lim_n W^n_\infty = \sum_{s<\infty} \lim_n [(H_s - H^1_s)(\Delta X_s, \Delta Y_s) 1_{\{|\Delta X_s| > \frac{1}{n}\}}]$$

$$= \sum_{s<\infty} (H_s - H^1_s)(\Delta X_s, \Delta Y_s) 1_{\{|\Delta X_s|>0\}}$$

$$= \sum_{s<\infty} (H_s - H^1_s)(\Delta X_s, \Delta Y_s).$$

The limit also holds in L^1, since for each n

$|W^n_\infty| \le 2k\int^\infty |d[X, Y]_s|$. Then by Theorem 3,

$$E[((H^1 \cdot X)_\infty + Z_\infty, Y_\infty)]$$

$$= E[((H^1 \cdot X)_\infty, Y_\infty)] + \lim_n E[(B^n_\infty - (B^n)^p_\infty, Y_\infty)]$$

$$= E[\int^\infty H^1_s d[X, Y]_s] + \lim_n E[\int_{]0,\infty]} \Delta Y_s dB^n_s]$$

$$= E[\int^\infty H^1_s d[X, Y]_s] + \lim_n E[\sum_{0<s<\infty} (H_s - H^1_s)(\Delta Y_s, \Delta X_s) 1_{\{|\Delta X_s| > \frac{1}{n}\}}]$$

$$= E[\int^\infty H^1_s d[X, Y]_s] + \lim_n E[\sum_{s<\infty} (H_s - H^1_s)(\Delta X_s, \Delta Y_s) 1_{\{|\Delta X_s| > \frac{1}{n}\}}]$$

$$= E[\int^\infty H^1_s d[X, Y]_s] + E[\sum_{s<\infty} (H_s - H^1_s)(\Delta X_s, \Delta Y_s)]$$

$$= E[\int^\infty H^1_s d[X, Y]_s] + E[\int^\infty (H_s - H^1_s) d[X, Y]_s]$$

$$= E[\int^\infty H_s d[X, Y]_s].$$

Hence, $H \underset{c}{\cdot} X = H^1 \cdot X + Z$, by Theorem III.1.2.

(b) Suppose H is E-valued, bounded, and optional and X is real. We must show that for every real square integrable martingale Y,

$$E[((H^1 \cdot X)_\infty + Z_\infty)Y_\infty] = E[\int^\infty H_s d[X, Y]_s].$$

The same argument in (a) works in this case provided we can show

$$E[(B^n_\infty - (B^n)^p_\infty)Y_\infty] = E[\int_{]0,\infty]} \Delta Y_s dB^n_s].$$

(Here, Theorem 3 does not apply since B^n is E-valued and Y is real.)

First assume Y is bounded. Write $B = B^n$. Since stopping and compensation commute, we may localize and assume that B is of integrable variation. Then by the definition of dual projection, we have

$$\begin{aligned} E[(B_\infty - B^p_\infty)Y_\infty] &= E[\int^\infty Y_\infty dB_s] - E[\int^\infty Y_\infty dB^p_s] \\ &= E[\int^\infty {}^0Y_\infty dB_s] - E[\int^\infty {}^pY_\infty dB_s] \\ &= E[\int^\infty Y_s dB_s] - E[\int^\infty Y_{s-} dB_s] \\ &= E[\int^\infty \Delta Y_s dB_s] \\ &= E[\int_{]0,\infty]} \Delta Y_s dB_s]. \end{aligned}$$

The last equality holds since $B_0 = 0$ (see Definition 5).

Assuming now that B is only locally integrable, let (T_n) be a sequence of stopping times which increase to infinity such that B^{T_n} is of integrable variation. Then $B^{T_n} - (B^{T_n})^p = (B - B^p)^{T_n}$ converges in M_E^2 to $B - B^p$. Hence,

$$E[(B_\infty - B_\infty^p)Y_\infty] = \lim_n E[(B_\infty^{T_n} - (B^{T_n})_\infty^p)Y_\infty]$$

$$= \lim_n E[\int_{]0,\infty]} \Delta Y_s dB_s^{T_n}]$$

$$= \lim_n E[\int_{]0,T_n]} \Delta Y_s dB_s]$$

$$= E[\int_{]0,\infty]} \Delta Y_s dB_s].$$

Hence, for bounded Y we have

$$E[((H^1 \cdot X)_\infty + Z_\infty)Y_\infty] = E[\int^\infty H_s d[X, Y]_s]. \tag{7.1}$$

Then for Y in M^2, we may obtain a sequence of bounded martingales (Y^n) which converge to Y in M^2. Then for each n,

$$E[((H^1 \cdot X)_\infty + Z_\infty)Y_\infty^n] = E[\int^\infty H_s d[X, Y^n]_s]$$

Letting n tend to infinity, we obtain the expression in (7.1), since

$$E[|((H^1 \cdot X)_\infty + Z_\infty)(Y_\infty^n - Y_\infty)|] \leq \|(H^1 \cdot X)_\infty + Z_\infty\|_{L_E^2} \|Y_\infty^n - Y_\infty\|_{L^2}$$

and

$$E[\int^{\infty}|H_s||d[X, Y^n-Y]_s|] \le \|H\|_X\|Y^n - Y\|_{L^2}.$$

Thus, Definition 5 offers a new definition of the optional integral, in terms of the predictable integral, in all settings.

We point out that in the definition, we may always take $H^1 = {}^pH$. If H is cadlag and $H_0 = 0$, we may take $H^1 = H_-$.

We shall now give a characterization of $H\underset{c}{\cdot}X$ which will be useful in proving its properties.

8 THEOREM. Let H be a bounded, optional process and let X be in M^2_E. $H\underset{c}{\cdot}X$ is the unique real square integrable martingale such that for every Y in M^2,

$$E[(H\underset{c}{\cdot}X)_\infty Y_\infty] = E[\int^{\infty}H_s dB^Y_s]$$

where B^Y is the $E^* = E$ valued process of integrable variation given by

$$(\phi, B^Y_t(w)) = [Y, (\phi, X)]_t(w), \ \phi \in E.$$

Proof: The uniqueness is obvious for if U were another such square integrable martingale satisfying the theorem, then $E[(H\underset{c}{\cdot}X)_\infty Y_\infty] = E[U_\infty Y_\infty]$ for every Y in M^2. Hence, $(H\underset{c}{\cdot}X)_\infty = U_\infty$ a.s. and U is indistinguishable from $H\underset{c}{\cdot}X$.

In Theorem II.1.5, we saw that if we define

$$A^Y = \langle Y, (\cdot, X)\rangle,$$

then A^Y is of integrable variation, and for every K in $L^2_p(X)$

$$E[(K\cdot X)_\infty Y_\infty] = E[\int^\infty K_s dA^Y_s].$$

By replacing sharp brackets by square brackets, one may use the same proof to show that B^Y is of integrable variation and

$$E[(K\cdot X)_\infty Y_\infty] = E[\int^\infty K_s dB^Y_s]. \tag{8.1}$$

Then writing $H\underset{c}{\cdot}X = H^1\cdot X + M^2 - \lim_n(B^n - (B^n)^p)$ where $B^n_t = \sum_{s\leq t} (H_s - H^1_s, \Delta X_s) 1_{\{|\Delta X_s| > \frac{1}{n}\}}$ and H^1 is appropriately chosen, we have by (8.1) and Theorem 3 that

$$E[(H\underset{c}{\cdot}X)_\infty Y_\infty] = E[(H^1\cdot X)_\infty Y_\infty] + \lim_n E[(B^n_\infty - (B^n)^p_\infty)Y_\infty] \tag{8.2}$$

$$= E[\int^\infty H^1_s dB^Y_s] + \lim_n E[\int_{]0,\infty]} \Delta Y_s dB^n_s].$$

However,

$$\int_{]0,\infty]} \Delta Y_s dB^n_s = \sum_{0<s<\infty} \Delta Y_s \Delta B^n_s \tag{8.3}$$

$$= \sum_{0<s<\infty} \Delta Y_s (H_s - H^1_s, \Delta X_s) 1_{\{|\Delta X_s| > \frac{1}{n}\}}$$

$$= \sum_{s<\infty} \Delta Y_s (H_s - H^1_s, \Delta X_s) 1_{\{|\Delta X_s| > \frac{1}{n}\}}.$$

On the other hand, for every f in E

$$\begin{aligned}(f, \Delta B^Y) &= \Delta[Y, (f, X)] \\ &= \Delta Y \Delta(f, X) \\ &= \Delta Y(f, \Delta X);\end{aligned}$$

hence, since $H - H^1$ is thin

$$\begin{aligned}\int^\infty (H_s - H_s^1) dB_s^Y &= \sum_{s<\infty} (H_s - H_s^1, \Delta B_s^Y) \\ &= \sum_{s<\infty} \Delta Y_s (H_s - H_s^1, \Delta X_s).\end{aligned}$$

Moreover,

$$\begin{aligned}\sum_{s<\infty} |\Delta Y_s (H_s - H_s^1, \Delta X_s)| &\le \sum_{s<\infty} |H_s - H_s^1| |\Delta B_s^Y| \\ &= \int^\infty |H_s - H_s^1| |dB_s^Y|.\end{aligned} \tag{8.4}$$

The final expression is integrable, hence finite almost surely, since $H - H^1$ is bounded and B^Y is of integrable variation. Hence for a.e. w, the sum

$$\int_{]0,\infty]} \Delta Y_s dB_s^n = \sum_{s<\infty} \Delta Y_s (H_s - H_s^1, \Delta X_s) 1_{\{|\Delta X_s| > \frac{1}{n}\}}$$

is absolutely convergent uniformly in n. So by the Moore Lemma, we have a.s.

$$\lim_n \int_{]0,\infty]} \Delta Y_s dB_s^n = \sum_{s<\infty} \lim_n [\Delta Y_s (H_s - H_s^1, \Delta X_s) 1_{\{|\Delta X_s| > \frac{1}{n}\}}]$$

$$= \sum_{s<\infty} \Delta Y_s (H_s - H_s^1, \Delta X_s)$$

$$= \int^\infty (H_s - H_s^1) dB_s^Y.$$

Moreover, the limit holds in L^1 by dominated convergence since for every n

$$\left| \int_{]0,\infty]} \Delta Y_s dB_s^n \right| \leq \int^\infty |H_s - H_s^1| \, |dB_s^Y|,$$

by (8.3) and (8.4).

Hence, (8.2) becomes

$$E[(H_{\dot{c}}X)_\infty Y_\infty] = E[\int^\infty H_s^1 dB_s^Y] + E[\int^\infty (H_s - H_s^1) dB_s^Y]$$

$$= E[\int^\infty H_s dB_s^Y]$$

and the Theorem is proved.

9 COROLLARY. Let H be bounded and optional and let X be in M_E^2. For every Y in M^2, the process $M = [H_{\dot{c}}X, Y] - H \cdot B^Y$ is a uniformly integrable martingale which is zero at 0, where B^Y is defined by $B^Y = [Y, (\cdot, X)]$.

Proof: We note that for every stopping time T, $(B^Y)^T = B^{Y^T}$; hence,

$$E[M_T] = E[[H_{\dot{c}}X, Y]_T] - E[\int^T H_s dB_s^Y]$$

$$= E[[H\underset{c}{\cdot}X, Y^T]_\infty] - E[\int^\infty H_s d(B^Y)^T_s]$$

$$= E[(H\underset{c}{\cdot}X)_\infty Y^T_\infty] - E[\int^\infty H_s dB^{Y^T}_s]$$

$$= 0,$$

by Theorem 8. Hence, M is a uniformly integrable martingale. Since $B^Y_0 = Y_0(\cdot, X_0)$, we have

$$M_0 = (H\underset{c}{\cdot}X)_0 Y_0 - H_0 B^Y_0$$

$$= (H\underset{c}{\cdot}X)_0 Y_0 - Y_0(H_0, X_0).$$

Thus, to show $M_0 = 0$, it suffices to show that $(H\underset{c}{\cdot}X)_0 = (H_0, X_0)$. But this fact is obvious by Definition 5; for we have $B^n_0 - (B^n)^p_0 = 0$ and hence $Z_0 = L^2 - \lim\limits_n(B^n_0 - (B^n)^p_0) = 0$ a.s. Thus, $(H\underset{c}{\cdot}X)_0 = (H^1 \cdot X)_0 = (H^1_0, X_0) = (H_0, X_0)$.

Corollary 9 is one of many properties of the optional integral. We refrain from listing all of the properties here, for in the next section we shall extend $H\underset{c}{\cdot}X$ for locally bounded H and special semimartingales X. The properties of this extension will of course hold in particular for bounded H and square integrable X. We shall, however, give several more theorems in this section.

10 THEOREM. Let X be a square integrable martingale that is also a process of finite variation and let H be bounded and optional. Let H^1 be bounded and predictable such that $H^1_0 = H_0$ and $H - H^1$ is thin. Then,

(a) $H \underset{c}{\cdot} X = H^1 \cdot X + B - B^p$,

where $B_t = \sum_{s \leq t} (H_s - H_s^1, \Delta X_s) = \int^t (H_s - H_s^1) dX_s$, and

(b) $H \underset{c}{\cdot} X = (H_0, X_0) + A - A^p$,

where $A_t = \int^t H_s dX_s$.

Proof: We note that X is locally integrable and $X^p = X_0$. The processes A and B are then locally integrable, since H and $H - H^1$ are bounded by some constant k.

We let $B_t^n = \sum_{s \leq t} (H_s - H_s^1, \Delta X_s) 1_{\{|\Delta X_s| > \frac{1}{n}\}}$. By Definition 5, it suffices to show $B - B^p$ is in M^2 and $B - B^p = M^2 - \lim_n (B^n - (B^n)^p)$. As in Theorem 4, we have

$$\begin{aligned} E[[B-B^p, B-B^p]_\infty] &\leq 4E[[B, B]_\infty] \\ &= 4E[\sum_{s<\infty} |\Delta B_s|^2] \\ &= 4E[\sum_{s<\infty} (H_s - H_s^1, \Delta X_s)^2] \\ &\leq 16k^2 E[\sum_{s<\infty} \Delta[X, X]_s] \\ &\leq 16k^2 E[[X, X]_\infty] \\ &< \infty. \end{aligned}$$

Hence, $B - B^p$ is in M^2. Moreover,

$$(10.1) \qquad \|(B - B^p) - (B^n - (B^n)^p)\|^2_{M^2}$$

$$= E[[B - B^n - (B - B^n)^p,\ B - B^n - (B - B^n)^p]_\infty]$$

$$\leq 4E[[B - B^n,\ B - B^n]_\infty]$$

$$= 4E[\sum_{s<\infty} |\Delta(B - B^n)_s|^2]$$

$$= 4E[\sum_{s<\infty} (H_s - H^1_s,\ \Delta X_s)^2 1_{\{|\Delta X_s| \leq \frac{1}{n}\}}]$$

$$\leq 16k^2 E[\sum_{s<\infty} |\Delta X_s|^2 1_{\{|\Delta X_s| \leq \frac{1}{n}\}}].$$

Then since $[X\ X]_\infty$ is integrable and

$$\sum_{s<\infty} |\Delta X_s|^2 = \sum_{s<\infty} \Delta[X,\ X]_s \leq [X,\ X]_\infty,$$

the sum $\sum_{s<\infty} |\Delta X_s|^2 1_{\{|\Delta X_s| \leq \frac{1}{n}\}}$ converges uniformly in n, for almost every w. Hence,

$$\lim_n \sum_{s<\infty} |\Delta X_s|^2 1_{\{|\Delta X_s| \leq \frac{1}{n}\}} = \sum_{s<\infty} |\Delta X_s|^2 1_{\{|\Delta X_s| \leq 0\}}$$

$$= 0.$$

By dominated convergence, the limit holds in L^1. Hence, taking the limit in (10.1) as n tends to infinity, we obtain

$$B - B^p = M^2 - \lim_n (B^n - (B^n)^p),$$

and (a) is proved. Then since $X^p = X_0$, we have

$$\left(\int^{\cdot} H_s^1 dX_s\right)_t^p = \int^t H_s^1 dX_s^p = (H_0^1, X_0) = (H_0, X_0).$$

Hence,

$$H^1 \cdot X + B - B^p = \int^t H_s^1 dX_s + \int^t (H_s - H_s^1) dX_s - \left(\int^{\cdot} (H_s - H_s^1) dX_s\right)_t^p$$

$$= A - A^p + (H_0, X_0).$$

11 THEOREM. Let H be bounded and optional and let X be in M_E^2. Let H^1 be bounded and predictable such that $H_0^1 = H_0$ and $H - H^1$ is thin. Then

$$|H \underset{c}{\cdot} X|_{M^2} \leq \left(E\left[\int^\infty |H_s^1|^2 d\langle X, X\rangle_s\right]\right)^{1/2}$$

$$+ 2\left(E\left[\int^\infty |H_s - H_s^1|^2 d[X, X]_s\right)^{1/2}.$$

Proof: Let B_t^n be defined as in Definition 5. Then

$$(11.1) \qquad |H \underset{c}{\cdot} X|_{M^2} \leq |H^1 \cdot X|_{M^2} + \lim_n \|B^n - (B^n)^p\|_{M^2}.$$

By Theorem II.2.2 (a), we have

$$|H^1 \cdot X|_{M^2} \leq |H|_X = \left(E\left[\int^\infty |H_s^1|^2 d\langle X, X\rangle_s\right]\right)^{1/2}.$$

By line (4.1) of Theorem 4, we have for every n

$$\begin{aligned}\|B^n - (B^n)^p\|_{M^2} &= (E[[B^n-(B^n)^p,\ B^n-(B^n)^p]_\infty])^{1/2}\\ &\leq 2(E[\sum_{s<\infty} |(H_s-H_s^1,\ \Delta X_s)|^2])^{1/2}\\ &\leq 2(E[\sum_{s<\infty} |H_s - H_s^1|^2|\Delta X_s|^2])^{1/2}\\ &= 2(E[\int^\infty |H_s-H_s^1|^2 d[X,\ X]_s])^{1/2}.\end{aligned}$$

Hence, the theorem follows by line (11.1).

12 THEOREM. Let (H^k) be a uniformly bounded sequence of optional processes converging pointwise to H. Then for any X in M_E^2, $H^k \underset{c}{\cdot} X$ converges to $H \underset{c}{\cdot} X$ in M^2.

Proof: Let H^1 be chosen for H according to Definition 5. We choose, for each k, a bounded, predictable process J^k such that $J_0^k = H_0^k$ and $H^k - J^k$ is thin. We may also assume that (J^k) are uniformly bounded, by Lemma 1. Let $A = \{\lim_k J^k \neq H^1\}$. Then A is a predictable set. Let

$L^k = J^k 1_{A^c} + H^1 1_A$ for each k. L^k is still bounded and predictable.

Since $J_0^k = H_0^k$ converges to $H_0 = H_0^1$, we have $L_0^k = J_0^k = H_0^k$ for each k. Moreover, since

$$A \subseteq (\bigcup_k \{J^k \neq H^k\}) \cup \{H \neq H^1\},$$

A is thin. Then since

$$\{H^k \neq L^k\} \subseteq \{H^k \neq J^k\} \cup A,$$

we have that $H^k - L^k$ is still thin. Thus, $H^k \underset{c}{\cdot} X$ may be defined in terms of L^k, by Theorem 6. Moreover, (L^k) are uniformly bounded and (L^k) converges pointwise to H^1. Let

$$B_t^n = \sum_{s \leq t} (H_s - H_s^1, \Delta X_s) 1_{\{|\Delta X_s| > \frac{1}{n}\}}$$

and

$$B_t^{nk} = \sum_{s \leq t} (H_s^k - L_s^k, \Delta X_s) 1_{\{|\Delta X_s| > \frac{1}{n}\}}.$$

Then

$$H \underset{c}{\cdot} X = H^1 \cdot X + M^2 - \lim_n (B^n - (B^n)^p)$$

and

$$H^k \underset{c}{\cdot} X = L^k \cdot X + M^2 - \lim_n (B^{nk} - (B^{nk})^p)$$

and

$$\begin{aligned} H \underset{c}{\cdot} X - H^k \underset{c}{\cdot} X &= (H^1 - L^k) \cdot X + M^2 - \lim_n (B^n - B^{nk} - (B^n - B^{nk})^p) \\ &= (H - H^k) \underset{c}{\cdot} X, \end{aligned}$$

since $H^1 - L^k$ satisfies Definition 5 for $H - H^k$. Thus, by Theorem 11,

$$\begin{aligned} \|H \underset{c}{\cdot} X - H^k \underset{c}{\cdot} X\|_{M^2} &= \|(H - H^k) \underset{c}{\cdot} X\|_{M^2} \\ &\leq (E[\int^\infty |H_s^1 - L_s^k|^2 d\langle X, X\rangle_s])^{1/2} \\ &\quad + 2(E[\int^\infty |(H_s - H_s^k) - (H_s^1 - L_s^k)|^2 d[X, X]_s])^{1/2}. \end{aligned}$$

The theorem follows by dominated convergence, since $(H^1 - L^k)$ are uniformly bounded and converge pointwise to 0 and $((H - H^k) - (H^1 - L^k))$ also are uniformly bounded and converge to 0.

13 THEOREM. Let (X^n) be a sequence of square integrable martingales converging to X in M_E^2. If H is bounded and optional, then $(H \underset{c}{\cdot} X^n)$ converges to $H \underset{c}{\cdot} X$ in M^2.

Proof: We choose H^1 according to Definition 5. Using Theorem 3.1 to follow, we have $H \underset{c}{\cdot} X - H \underset{c}{\cdot} X^n = H \underset{c}{\cdot} (X - X^n)$. Then by Theorem 11, we have, taking H and H^1 to be bounded by k, that

$$\begin{aligned} \| H \underset{c}{\cdot} X - H \underset{c}{\cdot} X^n \|_{M^2} &= \| H \underset{c}{\cdot} (X - X^n) \|_{M^2} \\ &\leq k(E[\langle X-X^n, X-X^n \rangle_\infty])^{1/2} \\ &\quad + 4k(E[[X-X^n, X-X^n]_\infty)^{1/2} \\ &= 5k(E[[X-X^n, X-X^n]_\infty)^{1/2} \\ &= 5k \| X - X^n \|_{M_E^2}. \end{aligned}$$

The result follows by letting n tend to infinity.

2. The Optional Integral With Respect to a Special Semimartingale

In this section, we extend the integral defined in Section 1 to all locally bounded, optional processes H and all special semimartingales X. As in Section IV.3, we shall make this extension by pasting and piecing together the different parts of the integral. We continue to assume that both H and X are E-valued.

1 LEMMA. Let X be a local martingale of finite variation which is zero at 0 and let H be locally bounded and optional. We set $A_t = \int^t H_s dX_s$ and $N = A - A^p$. Then for every real local martingale Y, the process $[N, Y] - H\cdot B^Y$ is a local martingale which is zero at 0, where $B^Y = [Y, (\cdot, X)]$.

Proof: Since X is a local martingale of finite variation, it is locally integrable. Then A is also locally integrable since H is locally bounded. Moreover,

$$B_t^Y = \sum_{s\leq t} \Delta Y_s \Delta(\cdot, X)_s = \sum_{s\leq t} \Delta Y_s(\cdot, \Delta X_s);$$

and

$$(H\cdot B^Y)_t = \sum_{s\leq t} (H_s, \Delta B_s^Y) = \sum_{s\leq t} \Delta Y_s(H_s, \Delta X_s) = \sum_{s\leq t} \Delta Y_s \Delta A_s = [A, Y]_t.$$

Hence, $[N, Y] - H\cdot B^Y = -[A^p, Y]$ is a local martingale, by Theorem II.3.4 (taking E to be the real line). Since $X_0 = 0$, we have $A_0 = 0$; but then $A_0^p = 0$ and $[A^p, Y]_0 = 0$. The lemma is then proved.

2 THEOREM. Let X be a special semimartingale having canonical decomposition $X = X_0 + M + A$ and let H be a locally bounded, optional process. There exists a unique real semimartingale Z having decomposition $Z = (H_0, X_0) + W + H\cdot A$ such that for every real bounded martingale Y, the process $[W, Y] - H\cdot B^Y$ is a local martingale, which is zero at 0, where $B^Y = [Y, (\cdot, X)]$.

Z is called the optional (or compensated) stochastic integral of H with respect to X and we denote $W = H \underset{c}{\cdot} M$ and $Z = H \underset{c}{\cdot} X$. Moreover, if H is predictable, then $H \underset{c}{\cdot} X = H \cdot X$.

Proof: For the uniqueness, we must show the uniqueness of the local martingale W in the decomposition of Z. If U is another such local martingale, then for every real bounded martingale Y, $[W-U, Y]$ is a local martingale which is zero at 0. By taking E to be the real line in the proof of uniqueness in Theorem III.3.3, we have $W = U$.

An immediate consequence of the uniqueness is the following:

(2.1) If M and N are local martingales, zero at 0,
and $H \underset{c}{\cdot} (M + N)$, $H \underset{c}{\cdot} M$, and $H \underset{c}{\cdot} N$ are defined, then
$H \underset{c}{\cdot} (M + N) = H \underset{c}{\cdot} M + H \underset{c}{\cdot} N$.

To see this result, let Y be a real bounded martingale and let

$$B^Y = [Y, (\cdot, M+N)],$$

$$C^Y = [Y, (\cdot, M)],$$

and

$$D^Y = [Y, (\cdot, N)].$$

Then $B^Y = C^Y + D^Y$ and

$$[H \underset{c}{\cdot} M + H \underset{c}{\cdot} N, Y] - H \cdot B^Y = [H \underset{c}{\cdot} M, Y] - H \cdot C^Y + [H \underset{c}{\cdot} N, Y] - H \cdot D^Y$$

is a local martingale, zero at 0. Hence, $H \underset{c}{\cdot} M + H \underset{c}{\cdot} N = H \underset{c}{\cdot} (M + N)$, by uniqueness.

A second consequence of uniqueness is the following:

(2.2) If M is a local martingale, zero at 0, then for every stopping time T,

$$H1_{[0,T]} \underset{c}{\bullet} M = (H \underset{c}{\bullet} M)^T = H \underset{c}{\bullet} M^T.$$

To see this result, we let Y be a real bounded martingale and set

$$B^Y = [Y, (\cdot, M)],$$

and

$$\begin{aligned} C^Y &= [Y, (\cdot, M^T)] \\ &= [Y, (\cdot, M)]^T \\ &= (B^Y)^T. \end{aligned}$$

Then

$$\begin{aligned} [(H \underset{c}{\bullet} M)^T, Y] - H \bullet C^Y &= [H \underset{c}{\bullet} M, Y]^T - H \bullet (B^Y)^T \\ &= ([H \underset{c}{\bullet} M, Y] - H \bullet B^Y)^T \end{aligned}$$

is a local martingale, zero at 0, since $[H \underset{c}{\bullet} M, Y] - H \bullet B^T$ is a local martingale, zero at 0. Hence, $(H \underset{c}{\bullet} M)^T = H \underset{c}{\bullet} M^T$. Moreover, since also

$$[(H \underset{c}{\bullet} M)^T, Y] - H1_{[0,T]} \bullet B^Y = ([H \underset{c}{\bullet} M, Y] - H \bullet B^Y)^T,$$

we have $(H \underset{c}{\bullet} M)^T = H1_{[0,T]} \underset{c}{\bullet} M$. We now construct $H \underset{c}{\bullet} M$ for bounded H.

We write M = N + B, where N is a local martingale with bounded jumps, B is a local martingale of finite variation, and $N_0 = 0 = B_0$.

We let (T_n) be a sequence of stopping times which increase to infinity such that N^{T_n} is a square integrable martingale for each n. $H\underset{c}{\bullet}N^{T_n}$ is defined, by Definition 1.5, and satisfies the theorem, by Corollary 1.9. Moreover, by (2.2), we have

$$(H\underset{c}{\bullet}N^{T_{n+1}})^{T_n} = H\underset{c}{\bullet}N^{T_n}$$

and we may define $H\underset{c}{\bullet}N = \lim_n H\underset{c}{\bullet}N^{T_n}$ by pasting together, so that

$$(H\underset{c}{\bullet}N)^{T_n} = H\underset{c}{\bullet}N^{T_n}.$$

Let Y be a bounded martingale. We set $B^Y = [Y, (\cdot, N)]$ and $C^Y = [Y, (\cdot, N^{T_n})]$. Then $C^Y = (B^Y)^{T_n}$. Then for every n

$$([H\underset{c}{\bullet}N, Y] - H\cdot B^Y)^{T_n} = [(H\underset{c}{\bullet}N)^{T_n}, Y] - H\cdot(B^Y)^{T_n}$$

$$= [H\underset{c}{\bullet}N^{T_n}, Y] - H\cdot C^Y$$

is a local martingale, zero at 0. Thus, $[H\underset{c}{\bullet}N, Y] - H\cdot B^Y$ is also a local martingale, zero at 0, and $H\underset{c}{\bullet}N$ satisfies the theorem. Moreover, if H is predictable, then $H\underset{c}{\bullet}N^{T_n} = H\cdot N^{T_n}$ for all n, by Theorem 1.6. Then $H\underset{c}{\bullet}N = H\cdot N$, since, by pasting, $(H\underset{c}{\bullet}N)^{T_n} = (H\cdot N)^{T_n}$ for each n.

By Lemma 1, $H\underset{c}{\bullet}B$ is defined as $H\underset{c}{\bullet}B = A - A^p$, where $A_t = \int^t H_s dB_s$. Moreover, since $B^p = 0$, if H is predictable, then $A_t^p = \int^t H_s dB_s^p = 0$. Hence, $H\underset{c}{\bullet}B = A = H\bullet B$.

We may then let $H\underset{c}{\bullet}M = H\underset{c}{\bullet}N + H\underset{c}{\bullet}B$, by (2.1). Now suppose H is locally bounded. We choose stopping times (T_n) which increase to infinity such that $H1_{]0,T_n]}$ is bounded for each n. Then $W^n = H1_{]0,T_n]} \overset{\bullet}{c} M$ exists (for any local martingale M which is zero at 0). Then for any bounded martingale Y, $B_0^Y = Y_0(\cdot, M_0) = 0$; hence, $H1_{]0,T_n]} \cdot B^Y = H1_{[0,T_n]} \cdot B^Y$.

It follows that $H1_{[0,T_n]} \overset{\bullet}{c} M$ is defined and

$$H1_{[0,T_n]} \overset{\bullet}{c} M = H1_{]0,T_n]} \overset{\bullet}{c} M$$

for each n. Then, by (2.2), we have for each n that

$$(H1_{[0,T_{n+1}]} \overset{\bullet}{c} M)^{T_n} = H1_{[0,T_n]} \overset{\bullet}{c} M.$$

We may then define $H\underset{c}{\bullet}M = \lim_n H1_{[0,T_n]} \overset{\bullet}{c} M$ by pasting, so that for each n

$$(H\underset{c}{\bullet}M)^{T_n} = H1_{[0,T_n]} \overset{\bullet}{c} M.$$

Then for every bounded martingale Y,

$$([H \underset{c}{\bullet} M,\ Y] - H\bullet B^{Y})^{T_n} = [(H \underset{c}{\bullet} M)^{T_n},\ Y] - H1_{[0,T_n]} \bullet B^{Y}$$

$$= [H1_{[0,T_n]} \underset{c}{\bullet} M,\ Y] - H1_{[0,T_n]} \bullet B^{Y}$$

is a local martingale which is zero at 0; hence, $[H \underset{c}{\bullet} M,\ Y] - H\bullet B^{Y}$ is also a local martingale, zero at 0.

Moreover, if H is predictable, then $H1_{]0,T_n]}$ is still predictable; hence, for each n

$$(H \underset{c}{\bullet} M)^{T_n} = H1_{[0,T_n]} \underset{c}{\bullet} M$$

$$= H1_{]0,T_n]} \bullet M$$

$$= H1_{[0,T_n]} \bullet M$$

$$= (H\bullet M)^{T_n};$$

hence, $H \underset{c}{\bullet} M = H\bullet M$, by letting n tend to infinity.

3 REMARKS.

(a) We note that if we let $C^{Y} = [Y, (\cdot, X)]$, then

$$[H \underset{c}{\bullet} X,\ Y] - H\bullet C^{Y} = [H \underset{c}{\bullet} M,\ Y] - H\bullet B^{Y}$$

for every bounded martingale Y; hence, $[H \underset{c}{\bullet} X,\ Y] - H\bullet C^{Y}$ is also a local martingale which is zero at 0.

(b) We shall see in the following section that Theorem 2 actually holds for all real local martingales Y.

(c) Suppose H is bounded and X is in M_E^2. Then X has canonical decomposition $X = X_0 + (X - X_0)$. We choose H^1 according to Definition 1.5. We set

$$B_t^n = \sum_{s \le t} (H_s - H_s^1, \Delta X_s) 1_{\{|\Delta X_s| > \frac{1}{n}\}}$$

and

$$C_t^n = \sum_{s \le t} (H_s - H_s^1), \Delta(X - X_0)_s) 1_{\{|\Delta(X - X_0)_s| > \frac{1}{n}\}}.$$

Since $H_0 - H_0^1 = 0$ and $\Delta X = \Delta(X - X_0)$ on $]0, \infty]$, we have $B^n = C^n$. Then since $H^1 \cdot X = (H_0^1, X_0) + H^1 \cdot (X - X_0) = (H_0, X_0) + H^1 \cdot (X - X_0)$, we have $H \underset{c}{\cdot} X = (H_0, X_0) + H \underset{c}{\cdot} (X - X_0)$. Then, by Corollary 1.9, $W = H \underset{c}{\cdot} (X - X_0)$ satisfies Theorem 2; hence, $H \underset{c}{\cdot} X$ defined in Definition 1.5 agrees with $H \underset{c}{\cdot} X$ defined in Theorem 2.

3. Properties of the Optional Integral

We conclude this chapter with properties of the optional integral. These properties must be proved directly, unlike many of the corresponding properties in Chapters III and IV which could be proved weakly. Throughout the section, we prove the properties for a locally bounded, optional process H and a special semimartingale, having canonical decomposition $X = X_0 + M + A$. We note that, in particular, the results hold for bounded H and square integrable X.

1 THEOREM.

(a) If X and Z are special semimartingles, then

$$H\underset{c}{\cdot}(X + Z) = H\underset{c}{\cdot}X + H\underset{c}{\cdot}Z.$$

(b) If K is locally bounded, then $(H + K)\underset{c}{\cdot}X = H\underset{c}{\cdot}X + K\underset{c}{\cdot}X$.

Proof:

(a) We write $X = X_0 + M + A$ and $Z = Z_0 + N + B$. Then $X + Z$ has canonical decomposition $X + Z = (X_0 + Z_0) + (M + N) + (A + B)$. Clearly, $H\cdot(A + B) = H\cdot A + H\cdot B$. By line (2.1) of the proof of Theorem 2.2, we have $H\underset{c}{\cdot}(M + N) = H\underset{c}{\cdot}M + H\underset{c}{\cdot}N$, and (a) follows.

(b) We let $B^Y = [Y, (\cdot, M)]$, for every bounded martingale Y. Then

$$[H\underset{c}{\cdot}M + K\underset{c}{\cdot}M, Y] - (H + K)\cdot B^Y$$

$$= [H\underset{c}{\cdot}M, Y] - H\cdot B^Y + [K\underset{c}{\cdot}M, Y] - \quad B^Y$$

is a local martingale which is zero at 0. Hence, $H\underset{c}{\cdot}M + K\underset{c}{\cdot}M = (H + K)\underset{c}{\cdot}M$. Assertion (b) is then obvious, since $(H + K)\cdot A = H\cdot A + K\cdot A$.

2 THEOREM. For every stopping time T,

$$(H\underset{c}{\cdot}X)^T = H\underset{c}{\cdot}X^T = H1_{[0,T]}\underset{c}{\cdot}X.$$

Proof: Clearly, $(H\cdot A)^T = H\cdot A^T = H1_{[0,T]}\cdot A$. The theorem follows from line (2.2) in the proof of Theorem 2.2.

3 THEOREM. For every locally bounded H and special semimartingale $X = X_0 + M + A$,

(a) $(\ {}_cX)_0 = (H_0, X_0)$, and

(b) $\Delta(H \underset{c}{\cdot} X) = (H, \Delta M) - {}^p(H, \Delta M) + (H, \Delta A)$ on $]0, \infty[$.

Proof: The first assertion is obvious, since $H \underset{c}{\cdot} X = (H_0, X_0) + H \underset{c}{\cdot} M + H \cdot A$, where $(H \underset{c}{\cdot} M)_0 = 0 = (H \cdot A)_0$. It is clear that $\Delta(H \cdot A) = (H, \Delta A)$; hence, for the second assertion, it suffices to show $\Delta(H \underset{c}{\cdot} M) = (H, \Delta M) - {}^p(H, \Delta M)$ on $]0, \infty[$.

We first note that since M is a local martingale, ${}^p(\Delta M)$ exists and thus for every predictable stopping time T,

$$E(|\Delta M_T| 1_{(T<\infty)} | F_{T-}) < \infty \text{ a.s.}$$

Thus, for H bounded,

$$E(|(H_T, \Delta M_T)| 1_{(T<\infty)} | F_{T-}) < \infty \text{ a.s.},$$

for every predictable stopping time T. Hence, ${}^p(H, \Delta M)$ exists.

We let T be a stopping time, with $T > 0$, which is either predictable or totally inaccessible. We let U be a bounded, F_T-measurable random variable and $B_t = U1_{(T \le t)}$.

$Y = B - B^p$ is a square integrable martingale that is also a process of variation variation. Moreover, if T is predictable, then

$$Y = (U - E(U|F_{T-}))1_{(T \le t)}. \tag{3.1}$$

If T is totally inaccessible, then B^p is continuous and hence $\Delta Y = \Delta B$. These assertions on Y are well-known facts and may be found in Déllacherie and Meyer [4, VII.74].

We let $B^Y = [Y, (\cdot, M)] = \Delta Y_T(\cdot, \Delta M_T)1_{(T\leq t)}$. Then the process $C = [H\underset{c}{\cdot}M, Y] - B^Y$ is a local martingale which is zero at 0. We let (T_n) be a sequence of stopping times reducing both C and $H\underset{c}{\cdot}M$. Then,

$$C^{T_n} = [(H\underset{c}{\cdot}M)^{T_n}, Y] - H1_{[0,T_n]}\quad B^Y$$

$$= [H1_{[0,T_n]} \underset{c}{\cdot} M, Y] - H1_{[0,T_n]} \cdot B^Y.$$

We let $K^n = H1_{[0,T_n]}$ to simplify the notation. Then,

$$C_\infty^{T_n} = \sum_{s<\infty} \Delta[K^n\underset{c}{\cdot}M)_s \Delta Y_s - \int^\infty K_s^n dB_s^Y$$

$$= \Delta(K^n\underset{c}{\cdot}M)_T \Delta Y_T - \Delta Y_T(K_T^n, \Delta M_T).$$

Since C^{T_n} is a uniformly integrable martingale which is zero at 0, we have $E[C_\infty^{T_n}] = 0$; hence,

$$E[\Delta(K^n\underset{c}{\cdot}M)_T \Delta Y_T] = E[\Delta Y_T(K_T^n, \Delta M_T)]. \tag{3.2}$$

Assume now that T is predictable. By (3.1), the right hand side of (3.2) becomes

$$E[U(K_T^n, \Delta M_T)] - E[E(U|F_{T-})(K_T^n, \Delta M_T)] \tag{3.3}$$

$$= E[U(K^n_T,\ \Delta M_T)] - E[UE((K^n_T,\ \Delta M_T)|F_{T-})]$$

$$= E[U(K^n_T,\ \Delta M_T)] - E[U^p(K^n,\ \Delta M)_T]$$

$$= E[U((K^n_T,\ \Delta M_T) - {}^p(K^n,\ \Delta M)_T)]$$

and the left hand side of (3.2) becomes

(3.4)
$$E[\Delta(K^n \underset{c}{\cdot} M)_T U] - E[\Delta(K^n \underset{c}{\cdot} M)_T E(U|F_{T-})]$$

$$= E[\Delta(K^n \underset{c}{\cdot} M)_T U] - E[E(\Delta(K^n \underset{c}{\cdot} M)_T|F_{T-})U]$$

$$= E[\Delta(K^n \underset{c}{\cdot} M)_T U].$$

The last equality in (3.4) is obtained since $K^n \underset{c}{\cdot} M = (H \underset{c}{\cdot} M)^{T_n}$ is a uniformly integrable martingale and thus $E(\Delta(K^n \underset{c}{\cdot} M)_T|F_{T-}) = 0$ a.s., for predictable T. Since U is an arbitrary, bounded, F_T-measurable, random variable, we have, by comparing (3.2), (3.3), and (3.4), that

(3.5)
$$\Delta(K^n \underset{c}{\cdot} M)_T = (K^n_T,\ \Delta M_T) - {}^p(K^n,\ \Delta M)_T \text{ a.s.}$$

Now assume that T is totally inaccessible. Then $\Delta Y_T = \Delta B_T = U$; hence, (3.2) becomes

(3.6)
$$E[\Delta(K^n \underset{c}{\cdot} M)_T U] = E[U(K^n_T,\ \Delta M_T)].$$

Again since U is arbitrary, we obtain

(3.7)
$$\Delta(K^n{}_c M)_T = (K^n_T,\ \Delta M_T) \text{ a.s.}$$

We next observe that ΔM is thin; hence, $(K^n, \Delta M)$ is thin. Also, $(K^n, \Delta M) - {}^p(K^n, \Delta M)$ is thin. Hence, ${}^p(K^n, \Delta M)$ is thin. Since this process is predictable, the set $\{{}^p(K^n, \Delta M) \neq 0\}$ is indistinguishable from a countable union of disjoint graphs of predictable stopping times. Since T is totally inaccessible, it follows that

$$ {}^p(K^n, \Delta M)_T = 0 \text{ a.s.} $$

Hence, (3.7) becomes

$$ \Delta(K^n {}_c\!\cdot M)_T = (K^n_T, \Delta M_T) - {}^p(K^n, \Delta M)_T \text{ a.s.} \tag{3.8} $$

By (3.5) and (3.8), we have

$$ \Delta(K^n {}_c\!\cdot M)_T = (K^n_T, \Delta M_T) - {}^p(K^n, \Delta M)_T \text{ a.s.} $$

for any predictable or totally inaccessible stopping time T, such that $T > 0$. It follows that this equality holds for any arbitrary stopping time T, such that $T > 0$. Hence, by the cross-section theorem, we have on $]0, \infty]$

$$ \Delta(K^n {}_c\!\cdot M) = (K^n, \Delta M) - {}^p(K^n, \Delta M), $$

up to an evanescent set. However,

$$ \Delta(K^n {}_c\!\cdot M) = \Delta(H {}_c\!\cdot M)^{T_n} $$

and

$$(K^n, \Delta M) - {}^p(K^n, \Delta M) = 1_{[0,T_n]}(H, \Delta M) - {}^p(1_{[0,T_n]}(H, \Delta M))$$

$$= 1_{[0,T_n]}(H, \Delta M) - 1_{[0,T_n]}{}^p(H, \Delta M).$$

Letting n tend to infinity, we have on $]0, \infty[$ that $\Delta(H_{\dot{c}}M)$ and $(H, \Delta M) - {}^p(H, \Delta M)$ are indistinguishable.

To prove this result, we had assumed that H was bounded. If H is locally bounded, we let (T_n) be a sequence of stopping times, which increase to infinity, such that $H1_{]0,T_n]}$ is bounded. Then, since $M_0 = 0$, we have

$$H1_{]0,T_n]} \underset{c}{\cdot} M = H1_{[0,T_n]} \underset{c}{\cdot} M = (H_{\dot{c}}M)^{T_n}.$$

Hence, on $]0, \infty[$,

$$\Delta(H_{\dot{c}}M)^{T_n} = (H1_{]0,T_n]}, \Delta M) - {}^p(H1_{]0,T_n]}, \Delta M)$$

$$= 1_{]0,T_n]}[(H, \Delta M) - {}^p(H, \Delta M)].$$

We obtain the result by letting n tend to infinity.

Using the jump formula of Theorem 3, we shall now extend Theorem 2.2.

4 THEOREM. Let H be a locally bounded, optional process and let X be a special semimartingale with canonical decomposition $X = X_0 + M + A$.

Then for every real local martingale Y, the process $C = [H_c^{\bullet}M, Y] - H\cdot B^Y$ is a local martingale which is zero at 0, where $B^Y = [Y, (\cdot, M)]$.

Proof: Since $(H_c^{\bullet}M)_0 = 0$ and $M_0 = 0$, we have $[H_c^{\bullet}M, Y-Y_0] = [H_c^{\bullet}M, Y]$ and $[Y-Y_0, (\cdot, M)] = [Y, (\cdot, M)]$; hence, we may assume $Y_0 = 0$. We assume first that H is bounded.

We decompose $Y = N + P$, where N is a local martingale with bounded jumps, P is a local martingale of finite variation, and $N_0 = 0 = P_0$. Then,

$$C = [H_c^{\bullet}M, N] - H\cdot B^N + [H_c^{\bullet}M, P] - H\cdot B^P.$$

We let (T_n) be stopping times which increase to infinity such that N^{T_n} is a bounded martingale. Then since $(B^N)^{T_n} = B^{N^{T_n}}$, we have for each n

$$([H_c^{\bullet}M, N] - H\cdot B^N)^{T_n} = [H_c^{\bullet}M, N^{T_n}] - H\cdot B^{N^{T_n}}$$

is a local martingale which is zero at 0, by Theorem 2.2. Hence, $[H_c^{\bullet}M, N] - H\cdot B^N$ is also a local martingale, zero at 0.

Since P is a process of finite variation, we have

$$B_t^P = \sum_{s\le t} \Delta P_s(\cdot, \Delta M_s)$$

and thus

$$[H_c^{\bullet}M,\ P] - H\bullet B^P = \sum_{s\le t} \Delta(H_c^{\bullet}M)_s \Delta P_s - \sum_{s\le t} \Delta P_s (H_s,\ \Delta M_s)$$

$$= \sum_{0<s\le t} {}^{P}(H,\ \Delta M)_s \Delta P_s,$$

by the jump formula of Theorem 3. However, since $P_0 = 0$, we may include 0 in the last sum. Thus, we must show that

$$\sum_{s\le t} {}^{P}(H,\ \Delta M)_s \Delta P_s$$

is a local martingale.

We decompose M in the same manner as Y. $M = W + Z$, where W is a local martingale with bounded jumps, Z is a local martingale of finite variation, and $W_0 = 0 = Z_0$.

Since $(H,\ \Delta W)$ and $(H,\ \Delta W) - {}^{P}(H,\ \Delta W)$ are thin, we have ${}^{P}(H,\ \Delta W)$ is also thin. Moreover, $(H,\ \Delta W)$ is bounded; hence, ${}^{P}(H,\ \Delta W)$ is bounded. Hence, by Lemma 1.2,

$$\sum_{s\le t} {}^{P}(H,\ \Delta W)_s \Delta P_s = {}^{P}(H,\ \Delta W)\bullet P,$$

the predictable stochastic integral, which is a local martingale, zero at 0, since P is a local martingale, zero at 0.

On the other hand, since P is of finite variation, $\int^t dP_s$ is locally integrable; hence, since H is bounded, $H\bullet P = \int^t H_s dP_s$ is also locally integrable. Thus, $(H\bullet P)^P$ exists and $\Delta(H\bullet P)^P = {}^{P}(\Delta(H\bullet P))$ $= {}^{P}(H,\ \Delta P)$. Thus,

$$\sum_{s\leq t} {}^{p}(H,\ \Delta P)_s \Delta P_s = [(H\cdot P)^p,\ P]$$

is a local martingale, by Theorem II.3.4. Hence,

$$\sum_{s\leq t} {}^{p}(H,\ \Delta M)_s \Delta P_s$$

is also a local martingale.

We have thus proved the theorem for bounded H. If H is locally bounded, we let (T_n) be stopping times which increase to infinity such that $H1_{]0,T_n]}$ is bounded. Then

$$\begin{aligned} C^{T_n} &= [H1_{[0,T_n]} \underset{c}{\cdot} M,\ Y] - H1_{[0,T_n]} \cdot B^Y \\ &= [H1_{]0,T_n]} \underset{c}{\cdot} M,\ Y] - H1_{]0,T_n]} \quad B^Y \end{aligned}$$

is a local martingale, zero at 0, for each n. The result then also holds for C.

5 THEOREM. Let (H^k) be a sequence of optional E-valued processes converging pointwise to H, and assume that (H^k) are uniformly bounded in norm by a locally bounded process K. Then $(H^k \underset{c}{\cdot} X)$ converges locally in S^1 to $H \underset{c}{\cdot} X$.

Proof: The proof is similar to Theorem IV.4.6. X has canonical decomposition $X = X_0 + M + A$. We then decompose M as $M = N + B$, where

N is a locally square integrable local martingale, B is a local martingale of finite variation, and $N_0 = 0 = B_0$.

As in Theorem IV.4.6, we choose stopping times (T_n) which increase to infinity such that for each n

(i) N^{T_n} is a square integrable martingale,

(ii) $K1_{]0,T_n]}$ is bounded,

(iii) $\int^{T_n} |K_s||dB_s|$ is integrable,

(iv) $\int^{T_n} |K_s||dA_s|$ is integrable, and

(v) $|K_0|X_0 1_{(T_n>0)}$ is integrable.

We now fix n and denote $T = T_n$.

By Theorem 2 and the fact that $N_0 = 0$, we have

$$(H \underset{c}{\bullet} N)^T = H1_{[0,T]} \underset{c}{\bullet} N^T = H1_{]0,T]} \underset{c}{\bullet} N^T.$$

Since $H1_{]0,T]}$ is bounded by $K1_{]0,T]}$, which is simply bounded, $H1_{]0,T]} \underset{c}{\bullet} N^T$ is a square integrable martingale. Thus,

$$\| H1_{]0,T]} \underset{c}{\bullet} N^T \|_{H^1} \leq \| H1_{]0,T]} \underset{c}{\bullet} N^T \|_{H^2} < \infty.$$

If we let $C_t = \int^t H_s dB_s^T$, then, by Lemma 2.1, $H \underset{c}{\bullet} B^T = C - C^p$. Hence, $H \underset{c}{\bullet} B^T$ is a local martingale that is also a process of finite variation. By Theorem II.5.2, applied to real processes, we have

$$\|H\cdot_c B^T\|_{H^1} \le E[\int^\infty |d(H\cdot_c B^T)_s|]$$

$$\le E[\int^\infty |dC_s|] + E[\int^\infty |dC^p_s|].$$

We saw in the proof of Corollary IV.3.6, that if C is an E-valued, in particular real-valued, process of locally integrable variation, then $E[\int^\infty |dC^p_s|] = E[\int^\infty |dC_s|]$. Hence,

$$\|H\cdot_c B^T\|_{H^1} \le 2E[\int^\infty |dC_s|]$$

$$= 2E[\int^T |H_s||dB_s|]$$

$$\le 2E[\int^T |K_s||dB_s|]$$

$$< \infty.$$

Since

$$(H\cdot_c X)^T 1_{(T>0)} = (H_0, X_0)1_{(T>0)} + (H\cdot_c N)^T + (H\cdot_c B)^T + (H\cdot A)^T$$

$$= (H_0, X_0)1_{(T>0)} + (H\cdot_c N)^T + H\cdot_c B^T + H\cdot A^T,$$

we have

(5.1) $$\|(H\cdot_c X)^T 1_{(T>0)}\|_{S^1}$$

$$\le \|(H_0, X_0)1_{(T>0)}\|_{L^1} + \|(H\cdot_c N)^T + (H\cdot_c B)^T\|_{H^1}$$

$$+ E[\int^\infty |H_s||dA^T_s|]$$

$$\leq \| |K_0| X_0 1_{(T>0)} \|_{L^1} + \| (H \underset{c}{\cdot} N)^T \|_{H^1} + \| H \underset{c}{\cdot} B^T \|_{H^1}$$

$$+ E[\int^T |K_s| |dA_s|]$$

$$< \infty.$$

Hence, $(H \underset{c}{\cdot} X)^T 1_{(T>0)}$ is in S^1. Likewise, $(H^k \underset{c}{\cdot} X)^T 1_{(T>0)}$ is in S^1 for each k. Replacing H by $L^k = H - H^k$ in (5.1), we obtain

$$\| (H \underset{c}{\cdot} X)^T 1_{(T>0)} - (H^k \underset{c}{\cdot} X)^T 1_{(T>0)} \|_{S^1}$$

$$= \| (L^k \underset{c}{\cdot} X)^T 1_{(T>0)} \|_{S^1}$$

$$\leq \| (L_0^k, X_0) 1_{(T>0)} \|_{L^1} + \| (L^k \underset{c}{\cdot} N)^T \|_{H^1} + \| L^k \underset{c}{\cdot} B^T \|_{H^1}$$

$$+ E[\int^T |L_s^k| |dA_s|].$$

By the dominated convergence theorem,

$$\lim_k \| (L_0^k, X_0) 1_{(T>0)} \|_{L^1} = 0,$$

$$\lim_k E[\int^T |L_s^k| |dA_s|] = 0,$$

and

$$\lim_k \| L^k \underset{c}{\cdot} B^T \|_{H^1} \leq 2 \lim_k E[\int^T |L_s^k| |dB_s|] = 0.$$

By Theorem 1.12, we have

$$\lim_k \|(L^k \underset{c}{\cdot} N)^T\|_{H^1} \leq \lim_k \|L^k 1_{]0,T]} \underset{c}{\cdot} N^T\|_{H^2}$$

$$= \lim_k \|L^k 1_{]0,T]} \underset{c}{\cdot} N^T\|_{M^2}$$

$$= 0.$$

Hence, $(H^k \underset{c}{\cdot} X)^T 1_{(T>0)}$ converges to $(H \underset{c}{\cdot} X)^T 1_{(T>0)}$ in S^1 and the theorem is proved.

6 THEOREM. Let H be locally bounded and optional and let X be a local martingale. Then $(H \underset{c}{\cdot} X)^c = H \underset{c}{\cdot} X^c$ and $(H \underset{c}{\cdot} X)^d = H \underset{c}{\cdot} X^d$.

Proof: Since $H \underset{c}{\cdot} X = (H \underset{c}{\cdot} X)^c + (H \underset{c}{\cdot} X)^d$ is a unique decomposition and $H \underset{c}{\cdot} X = H \underset{c}{\cdot} (X^c + X^d) = H \underset{c}{\cdot} X^c + H \underset{c}{\cdot} X^d$, it suffices to show $H \underset{c}{\cdot} X^c$ is continuous and $H \underset{c}{\cdot} X^d$ is purely discontinuous.

We have $(H \underset{c}{\cdot} X^c)_0 = (H_0, X_0^c) = 0$ and on $]0, \infty[$ $\Delta(H \underset{c}{\cdot} X\ ) = (H, \Delta X\ ) - {}^p(H, \Delta X^c) = 0$; hence, $H \underset{c}{\cdot} X^c$ is continuous.

Let Y be a continuous local martingale. Let $B^Y = [Y, (\cdot, X^d)]$. By Theorem II.3.9, we have for every f in E that

$$(f, B_t^Y) = [Y, (f, X^d)]_t = [fY, X^d]_t.$$

Since X^d is purely discontinuous and Y is continuous,

$$(f, B_t^Y) = \sum_{s \leq t} (\Delta fY, \Delta X_s^d) = \sum_{s \leq t} \Delta Y_s (f, \Delta X_s^d) = 0.$$

Hence, $B^Y = 0$. By Theorem 4, we have

$$[H_{\dot{c}}X^d, Y] - H \cdot B^Y = [H_{\dot{c}}X^d, Y]$$

is a local martingale which is zero at 0.

Hence, $H_{\dot{c}}X^d$ is orthogonal to every continuous local martingale. By Theorem III.5.8, $H_{\dot{c}}X^d$ is purely discontinuous, and the theorem is proved.

The following result is an immediate consequence of Definition 1.5 and relates the optional integral to the predictable stochastic integral when X is continuous.

7 THEOREM. Let X be a continuous special semimartingale and let H be locally bounded and optional. Then $H_{\dot{c}}X = H^1 \cdot X$, where H^1 is any locally bounded, predictable process such that $H - H^1$ is thin and $H_0^1 = H_0$.

Proof: Let $X = X_0 + M + A$ be the canonical decomposition. Since M is a local martingale, ${}^p(\Delta M)$ exists and ${}^p(\Delta M) = 0$. However, X is continuous, hence predictable, and A is predictable; thus, M is also predictable. Hence, $\Delta M = {}^p(\Delta M) = 0$ and M is continuous. Then A must also be continuous. Then since $H - H^1$ is thin, we have for each t

$$\int^t H_s dA_s = \int^t (H_s - H_s^1) dA_s + \int^t H_s^1 dA_s$$

$$= \sum_{s \leq t} (H_s - H_s^1, \Delta A_s) + \int^t H_s^1 dA_s$$

$$= \int^t H^1_s dA_s.$$

Hence, $H\cdot A = H^1\cdot A$.

Since M is continuous and zero at 0, there exists stopping times (T_n), which increase to infinity, such that M^{T_n} is a bounded, and hence square integrable, martingale for each n. Moreover, we may assume $H1_{]0,T_n]}$ and $H^1 1_{]0,T_n]}$ are bounded. Since M is zero at 0, we have for each n

$$(H\underset{c}{\cdot}M)^{T_n} = H1_{[0,T_n]} \underset{c}{\cdot} M^{T_n} = H1_{]0,T_n]} \underset{c}{\cdot} M^{T_n}$$

and

$$(H^1\cdot M)^{T_n} = H1_{[0,T_n]} \cdot M^{T_n} = H^1 1_{]0,T_n]} \cdot M^{T_n}.$$

Since $H1_{]0,T_n]} - H^1 1_{]0,T_n]}$ is still thin and M^{T_n} is a continuous square integrable martingale, we have, by Definition 1.5, that $H1_{]0,T_n]} \underset{c}{\cdot} M^{T_n} = H^1 1_{]0,T_n]} \cdot M^{T_n}$. Letting n tend to infinity, we obtain $H\underset{c}{\cdot}M = H^1\cdot M$. Hence,

$$\begin{aligned} H\underset{c}{\cdot}X &= (H_0, X_0) + H\underset{c}{\cdot}M + H\cdot A \\ &= (H^1_0, X_0) + H^1\cdot M + H^1\cdot A \\ &= H^1\cdot X. \end{aligned}$$

8 REMARKS.

(a) We consider the cases when

(i) H is real-valued and X is E-valued, and

(ii) H is E-valued and X is real.

We have noted, in Remarks 1.7, that definition 1.5 may be used to define $H \underset{c}{\cdot} X$ in the above two cases. Hence, Theorem 7, with the same proof, remains valid in both cases. Alternately, we have proven Theorem 7 when both H and X are E-valued. In particular, both H and X could be real-valued. Then, for every f in E, we have, in case (i), that

$$(f,\ H \underset{c}{\cdot} X) = H \underset{c}{\cdot} (f,\ X) = H^{1} \cdot (f,\ X) = (f,\ H^{1} \cdot X)$$

and in case (ii)

$$(f,\ H \underset{c}{\cdot} X) = (f,\ H) \underset{c}{\cdot} X = (f,\ H^{1}) \cdot X = (f,\ H^{1} \cdot X).$$

Thus, in the separable space E, the processes $H \underset{c}{\cdot} X$ and $H^{1} \cdot X$ are indistinguishable, in both cases.

This argument may be used to prove many properties in the previous two cases, once they are proved for both H and X being E-valued. However, we note that we are restricted to locally bounded H and the integrals of Chapters III and IV are defined for a different class, in general. In Section III.3, $H \underset{c}{\cdot} X$ was defined for H in $L_0(X)$. In Sections III.1 and IV.1, $H \underset{c}{\cdot} X$ was defined for H in $L_0^2(X)$.

(b) We note that if X is a continuous semimartingale (real or E-valued) and H is a bounded, optional process (real or E-valued), we may also unambiguously define $H \underset{c}{\cdot} X$ by $H \underset{c}{\cdot} X = H^{1} \cdot X$, where H^{1} is any bounded, predictable process such that $H - H^{1}$

is thin and $H^1_0 = H_0$. We refer to Lemma 1.1 for the existence of such an H^1.

We conclude with several "associativity" properties. We refer the reader to the previous results of Theorems II.2.7, III.4.4, and IV.4.4.

9 THEOREM. Let X be a special semimartingale, H locally bounded and optional, and K locally bounded and predictable.

(a) If H and X are E-valued and K is real, then

$$KH \underset{c}{\bullet} X = H \underset{c}{\bullet} (K \cdot X) = K \cdot (H \underset{c}{\bullet} X).$$

(b) If X is real and H and K are E-valued, then

$$(H, K) \underset{c}{\bullet} X = H \underset{c}{\bullet} (K \cdot X) = K \cdot (H \underset{c}{\bullet} X).$$

(c) If K and X are E-valued and H is real, then

$$HK \underset{c}{\bullet} X = H \underset{c}{\bullet} (K \cdot X) = K \cdot (H \underset{c}{\bullet} X).$$

Proof: In each case, we write $X = X_0 + M + A$, the canonical decomposition of X. The associativity clearly holds when integrating with respect to A, by the Stieltjes integral, and when forming the product with X_0; hence, we may assume $X = M$ is a local martingale which is zero at 0. We then write $M = N + P$, where N is a local martingale with bounded jumps, P is a local martingale of finite variation, and $N_0 = 0 = P_0$.

The proof for P is the same in each case. Let

$$B = \int^t H_s dP_s = H \cdot P,$$

$$C = \int^t K_s dP_s = K \cdot P,$$

and

$$D = \int^t K_s H_s dP_s = K \cdot B = H \cdot C.$$

Then,

$$KH \underset{c}{\cdot} P = D - D^p = (K \cdot B) - (K \cdot B)^p = K \cdot B - K \cdot B^p = K \cdot (B - B^p) = K \cdot (H \underset{c}{\cdot} P),$$

and $H \underset{c}{\cdot} (K \cdot P) = H \cdot C - (H \cdot C)^p = (K \cdot B) - (K \cdot B)^p = D - D^p = KH \underset{c}{\cdot} P.$

In case (b), we must simply replace KH by (K, H). For every stopping time T, since $N_0 = 0$, we have

$$(KH \underset{c}{\cdot} N)^T = KH1_{]0,T]} \underset{c}{\cdot} N^T,$$

$$(H \underset{c}{\cdot} (K \cdot N))^T = H1_{]0,T]} \underset{c}{\cdot} (K \cdot N)^T$$

$$= H1_{]0,T]} \underset{c}{\cdot} (K1_{]0,T]} \cdot N^T),$$

and

$$(K \cdot (H \underset{c}{\cdot} N))^T = K1_{]0,T]} \cdot (H \underset{c}{\cdot} N)^T$$

$$= K1_{]0,T]} \cdot (H1_{]0,T]} \underset{c}{\cdot} N^T).$$

Again, for case (b), we replace KH by (K, H). Hence, by localizing and stopping, we may assume that H and K are bounded and N is a square integrable martingale. We may then use Definition 1.5 to define all of the integrals, and we shall write X = N.

We begin with case (a).

Let H^1 be a bounded, E-valued, predictable process such that $H - H^1$ is thin and $H^1_0 = H_0$. Then

$$H \underset{c}{\cdot} (K \cdot X) = H^1 \cdot (K \cdot X) + M^2 - \lim_n (B^n - (B^n)^p)$$

where

$$B^n_t = \sum_{s \le t} (H_s - H^1_s, \Delta(K \cdot X)_s) 1_{\{|\Delta(K \cdot X)_s| > \frac{1}{n}\}}$$

$$= \sum_{s \le t} (H_s - H^1_s, K_s \Delta X_s) 1_{\{|K_s \Delta X_s| > \frac{1}{n}\}}.$$

Since $KH - KH^1$ is thin and KH^1 is predictable and bounded, we have

$$KH \underset{c}{\cdot} X = KH^1 \cdot X + M^2 - \lim_n (C^n - (C^n)^p)$$

where

$$C^n_t = \sum_{s \le t} (K_s H_s - K_s H^1_s, \Delta X_s) 1_{\{|\Delta X_s| > \frac{1}{n}\}}$$

$$= \sum_{s \le t} (H_s - H^1_s, K_s \Delta X_s) 1_{\{|\Delta X_s| > \frac{1}{n}\}}.$$

We have seen in part (d) of the proof of Theorem II.2.7 that $H^1 \cdot (K \cdot X) = H^1 K \cdot X = KH^1 \cdot X$, since K is real-valued. Hence, we must show that

$$M^2 - \lim_n (B^n - (B^n)^p) = M^2 - \lim_n (C^n - (C^n)^p).$$

We have, assuming that H, K, and H^1 are all bounded by k,

$$\|[B^n - (B^n)^p] - [C^n - (C^n)^p]\|^2_{M^2}$$

$$= E[[B^n - C^n - (B^n - C^n)^p,\ B^n - C^n - (B^n - C^n)^p]_\infty]$$

$$\leq 4E[[B^n - C^n,\ B^n - C^n]_\infty]$$

$$= 4E[\sum_{s<\infty} (\Delta(B^n - C^n)_s)^2]$$

$$= 4E[\sum_{s<\infty} (H_s - H^1_s,\ K_s\Delta X_s)^2 (1_{\{|K_s\Delta X_s| > \frac{1}{n}\}} - 1_{\{|\Delta X_s| > \frac{1}{n}\}})^2]$$

$$\leq 16k^2 E[\sum_{s<\infty} |K_s\Delta X_s|^2 (1_{\{|K_s\Delta X_s| > \frac{1}{n}\}} - 1_{\{|\Delta X_s| > \frac{1}{n}\}})^2]$$

$$= 16k^2 E[\sum_{s<\infty} (1_{\{|K_s\Delta X_s| > \frac{1}{n}\}} - 1_{\{|\Delta X_s| > \frac{1}{n}\}})^2 \Delta[K\cdot X,\ K\cdot X]_s]$$

$$= 16k^2 E[\int^\infty (1_{\{|K_s\Delta X_s| > \frac{1}{n}\}} - 1_{\{|\Delta X_s| > \frac{1}{n}\}})^2 d[K\cdot X,\ K\cdot X]^d_s].$$

By the dominated convergence theorem, this last expression converges, as n tends to infinity, to

$$16k^2 E[\int^\infty (1_{\{|K_s\Delta X_s|>0\}} - 1_{\{|\Delta X_s|>0\}})^2 d[K\cdot X,\ K\cdot X]^d_s]$$

$$= 16k^2 E\Big[\sum_{s<\infty} (1_{\{|K_s\Delta X_s|>0\}} - 1_{\{|\Delta X_s|>0\}})^2 |K_s\Delta X_s|^2\Big]$$

$$= 0,$$

since

$$(1_{\{|K_s\Delta X_s|>0\}} - 1_{\{|\Delta X_s|>0\}})^2 |K_s\Delta X_s|^2 = 0$$

for each s. Hence, $H\cdot_c(K\cdot X) = KH\cdot_c X$. We assert that $K\cdot(H\cdot_c X) = H\cdot(K\cdot_c X)$. Let Y be a real bounded martingale and let $C^Y = [Y, (\cdot, K\cdot X)]$. By Theorem II.4.8, we have for every f in E,

$$(f, C^Y) = [Y, (f, K\cdot X)]$$

$$= [Y, K\cdot(f, X)]$$

$$= [K\cdot Y, (f, X)].$$

Hence, $C^Y = B^{K\cdot Y}$, where B^Z is defined by $B^Z = [Z, (\cdot, X)]$, for every local martingale Z.

Again by Theorem II.4.8, we have $[K\cdot(H\cdot_c X), Y] = [H\cdot_c X, K\cdot Y]$; hence,

$$[K\cdot(H\cdot_c X), Y] - H\cdot C^Y = [H\cdot_c X, K\cdot Y] - H\cdot B^{K\cdot Y}$$

is a local martingale, zero at 0, by Theorem 4. Thus, by uniqueness, $K\cdot(H\cdot_c X) = H\cdot_c(K\cdot X)$, and (a) is proved. We next consider case (c).

Let H^1 be a real, bounded, predictable process such that $H - H^1$ is thin and $H_0^1 = H_0$. Then H^1K may be used to define $HK \underset{c}{\cdot} X$. We have

$$HK \underset{c}{\cdot} X = H^1K \cdot X + M^2 - \lim_n (B^n - (B^n)^p),$$

where

$$B_t^n = \sum_{s \le t} ((H_s - H_s^1)K_s, \Delta X_s) 1_{\{|\Delta X_s| > \frac{1}{n}\}},$$

and

$$H \underset{c}{\cdot} X = H^1 \cdot X + M^2 - \lim_n (C^n - (C^n)^p),$$

where

$$C_t^n = \sum_{s \le t} (H_s - H_s^1) \Delta X_s 1_{\{|\Delta X_s| > \frac{1}{n}\}}.$$

Since H^1 is real, we have, by Theorem II.2.7 (part (d) of the proof), that

$$K \cdot (H^1 \cdot X) = H^1 K \cdot X.$$

By Theorem 1.13, we have

$$K \cdot (M^2 - \lim_n (C^n - (C^n)^p)) = M^2 - \lim_n (K \cdot (C^n - (C^n)^p)).$$

However, $K \cdot (C^n - (C^n)^p) = K \cdot C^n - K \cdot (C^n)^p = K \cdot C^n - (K \cdot C^n)^p$, since K is predictable. We note then that $K \cdot C^n = B^n$. Hence,

$$K\cdot(H\underset{c}{\cdot}X) = H^1K\cdot X + M^2 - \lim_n(B^n - (B^n)^p)$$

$$= HK\underset{c}{\cdot}X.$$

But we assert that $K\cdot(H\underset{c}{\cdot}X) = H\underset{c}{\cdot}(K\cdot X)$.

Let Y be a real bounded martingale. Then by Theorem II.4.8,

$$[K\cdot(H\underset{c}{\cdot}X),\ Y] - H\cdot[K\cdot X,\ Y] = [H\underset{c}{\cdot}X,\ K\cdot Y] - H\cdot[X,\ K\cdot Y]$$

is a local martingale, zero at 0, by Theorem III.4.7. Hence, by uniqueness, $K\cdot(H\underset{c}{\cdot}X) = H\underset{c}{\cdot}(K\cdot X)$.

Lastly, we come to part (b). We again choose H^1 such that $H - H^1$ is thin and $H^1_0 = H_0$. Then $(H, K) - (H^1, K) = (H - H^1, K)$ is thin, and we have

$$(H, K)\underset{c}{\cdot}X = (H^1, K)\cdot X + M^2 - \lim_n(B^n - (B^n)^p),$$

where

$$B^n_t = \sum_{s\leq t}(H_s - H^1_s,\ K_s)\Delta X_s 1_{\{|\Delta X_s| > \frac{1}{n}\}}$$

$$= \sum_{s\leq t}(H_s - H^1_s,\ K_s\Delta X_s)1_{\{|\Delta X_s| > \frac{1}{n}\}},$$

and

$$H\underset{c}{\cdot}(K\cdot X) = H^1\cdot(K\cdot X) + M^2 - \lim_n(C^n - (C^n)^p),$$

where

$$C^n_t = \sum_{s\leq t}(H_s - H^1_s,\ \Delta(K\cdot X)_s)1_{\{|\Delta(K\cdot X)_s| > \frac{1}{n}\}}$$

$$= \sum_{s\leq t}(H_s - H^1_s,\ K_s\Delta X_s)1_{\{|K_s\Delta X_s| > \frac{1}{n}\}}.$$

By part (c) in the proof of Theorem II.2.7, we have $(H^1, K)\cdot X = H^1\cdot(K\cdot X)$. It follows, as in part (a), that

$$M^2 - \lim_n (B^n - (B^n)^p) = M^2 - \lim_n (C^n - (C^n)^p;$$

hence, $(H, K)\underset{c}{\cdot} X = H\underset{c}{\cdot}(K\cdot X)$. On the other hand,

$$H\underset{c}{\cdot} X = H^1\cdot X + M^2 - \lim_n (D^n - (D^n)^p),$$

where

$$D^n_t = \sum_{s\leq t} (H_s - H^1_s)\Delta X_s 1_{\{|\Delta X_s| > \frac{1}{n}\}}.$$

Again by part (c) of Theorem II.2.7, we have $K\cdot(H^1\cdot X) = (H^1, K)\cdot X$. By Theorem I.13, we have

$$K\cdot(M^2 - \lim_n (D^n - (D^n)^p)) = M^2 - \lim_n K\cdot(D^n - (D^n)^p).$$

We note that

$$K\cdot(D^n - (D^n)^p) = K\cdot D^n - (K\cdot D^n)^p$$

and

$$K\cdot D^n = B^n;$$

hence, $K\cdot(H\underset{c}{\cdot} X) = (H, K)\underset{c}{\cdot} X$, and the theorem is complete.

We refer the reader back to the proof of part (d) of Theorem II.2.7 and note that it is now complete, by taking H to be predictable in part (a) of Theorem 9.

CHAPTER VI
THE OPTIONAL STOCHASTIC INTEGRAL ON NUCLEAR SPACES

In this chapter, we shall apply the optional integrals defined in Chapters III, IV, and V to the new setting of a nuclear space. Since every nuclear space is (a subspace of) the projective limit of separable Hilbert spaces, the optional integrals will naturally extend to this new setting. We shall need several results on the theory of nuclear spaces, for which we shall refer to Pietsch [12] and Treves [14].

1. Notations

Throughout this chapter, we adopt the setting of Ustanel's paper [15], which develops the theory of the predictable stochastic integral on nuclear spaces.

Φ will denote a locally convex, reflexive, complete, bornological, nuclear space whose topological dual Φ^1 is complete and nuclear under its strong topology, denoted by Φ^1_b. Φ and Φ^1 are then projective limits of separable Hilbert spaces. We may then choose neighborhood bases $U_h(\Phi)$ and $U_h(\Phi^1_b)$, of Φ and Φ^1_b, respectively, such that the following propositions are all satisfied.

1 PROPOSITION.

(a) For every U in $U_h(\Phi)$, the quotient space $\Phi(U) = \Phi/N(U)$, where $N(U) = \{x \in \Phi: p_u(x) = 0\}$ and $p_u(x) = \inf\{\delta > 0: x \in \delta U\}$, is a separable Hilbert space under the norm induced by p_u.

(b) Φ has the least-fine topology such that each canonical mapping $k(U)$, from Φ onto $\Phi(U)$, is continuous.

(c) If V and U are in $U_h(\Phi)$ and $V \subseteq U$, then the canonical mapping $k(U, V)$ from $\Phi(V)$ onto $\Phi(U)$ is nuclear.

2 PROPOSITION.

(a) For every U in $U_h(\Phi_b^1)$, the quotient space $\Phi^1(U) = \Phi^1/N(U)$, where $N(U) = \{x \in \Phi^1: p_u(x) = 0\}$ and $p_u(x) = \inf\{\delta > 0: x \in \delta U\}$, is a separable Hilbert space under the norm induced by p_u.

(b) Φ_b^1 has the least-fine topology such that each canonical mapping $k(U)$, from Φ^1 onto $\Phi^1(U)$, is continuous.

(c) If V and U are in $U_h(\Phi_b^1)$ and $V \subseteq U$, then the canonical mapping $k(U, V)$ from $\Phi^1(V)$ onto $\Phi^1(U)$ is nuclear.

3 PROPOSITION. For every U in $U_h(\Phi)$, the dual of $\Phi(U)$ is (isometrically isomorphic) to the separable Hilbert space

$$\Phi^1[U^0] = \{x \in \Phi^1: x \in \delta U^0 \text{ for some } \delta > 0\}$$

having norm p_{u^0}, where U^0 is the polar of U. Moreover, $\Phi(U)$ and $\Phi^1[U^0]$ are put in duality by $\langle [b], x\rangle = \langle b, x\rangle$, where b is a representative of the equivalence class $[b]$.

Since Φ is reflexive, we also have the following result.

4 PROPOSITION. For every U in $U_h(\Phi_b^1)$, the dual of $\Phi^1(U)$ is (isometrically isomorphic) to the separable Hilbert space

$$\Phi[U^0] = \{x \in \Phi: x \in \delta U^0 \text{ for some } \delta > 0\}$$

having norm p_{u^0}, where U^0 is the polar of U. Moreover, $\Phi^1(U)$ and $\Phi[U^0]$ are put in duality by $\langle [b], x\rangle = \langle b, x\rangle$, where b is a representative of the equivalence class [b].

We point out that we may define $\Phi(U)$ (resp. $\Phi^1(U)$) and p_u for any closed, absolutely convex, zero neighborhood U of Φ (resp. Φ^1). Moreover, we may define $\Phi[V]$ (resp. $\Phi^1[V]$) for any closed, absolutely convex, bounded subset V of Φ (resp. Φ^1).

Lastly, we have

5 PROPOSITION.

(a) The set $K_h(\Phi) = \{U^0: U \in U_h(\Phi_b^1)\}$ is a fundamental system of (closed, absolutely convex) bounded sets in Φ.

(b) The set $K_h(\Phi^1) = \{U^0: U \in U_h(\Phi)\}$ is a fundamental system of (closed, absolutely convex) bounded sets in Φ^1.

2. Processes in Nuclear Space

In this section, we define, according to Ustanel [15], the concept of a projective system of stochastic processes in Φ^1. We shall also strengthen his result on the existence of a limit in Φ^1 of a projective system of square integrable martingales.

1 DEFINITION. The set $X = \{X^u: U \in U_h(\Phi_b^1)\}$ is called a projective system of semimartingales (resp. local martingales, martingales, processes of finite variation, etc.) if the following conditions hold:

(i) For every U in $U_h(\Phi_b^1)$, X^u is a $\Phi^1(U)$-valued semimartingale (resp. local martingale, martingale, process of finite variation, etc.).

(ii) If $V \subseteq U$ and $V \in U_h(\Phi_b^1)$, then $k(U, V)X^v$ and X^u are indistinguishable.

We point out that X^u is assumed to be cadlag in the strong topology of $\Phi^1(U)$, for every U in $U_h(\Phi_b^1)$.

2 DEFINITION. Let $X = \{X^u: U \in U_h(\Phi_b^1)\}$ be a projective system of semimartingales. We say X has a limit in Φ^1 if there exists a weakly adapted mapping $X^1: R^+ \times \Omega \to \Phi^1$ such that for every U in $U_h(\Phi_b^1)$, $k(U)X^1$ is a modification of X^u.

3 THEOREM. Let $M = \{M^u: U \in U_h(\Phi_b^1)\}$ be a projective system of square integrable martingales and let M^1 be a limit in Φ^1 of M. There exists a limit in Φ^1 of M, N^1, such that N^1 is strongly cadlag in Φ^1 and for every U in $U_h(\Phi_b^1)$, $k(U)N^1$ and M^u are indistinguishable.

Proof: Let $x \in \Phi$. Then $x \in \Phi[U^0]$, for some $U \in U_h(\Phi_b^1)$. By Proposition 1.4, we have for every t,

$$(3.1) \qquad \langle M_t^1, x\rangle = \langle k(U)M_t^1, x\rangle = \langle M_t^u, x\rangle \text{ a.s.}$$

Hence, $\langle M^1, x\rangle$ has a modification in M^2, the Hilbert space of real, (cadlag), square integrable martingales (in which indistinguishable processes are identified).

We define a mapping $T\colon \Phi \to M^2$ such that for every $x \in \Phi$, $T(x)$ is the modification of $\langle M^1, x\rangle$ which is in M^2. T is well-defined, since modifications in M^2 are indistinguishable by right continuity. Moreover, by (3.1), we have

(3.2) $T(x) = \langle M^u, x\rangle$, where U is any set in $U_h(\Phi_b^1)$ such that $x \in \Phi[U^0]$.

T is clearly linear, since $\langle M^1, \cdot\rangle$ is a linear mapping. We assert that T is continuous.

Suppose (x_n) converges to x in $\Phi[U^0]$, where $U \in U_h(\Phi_b^1)$. By (3.2), we have

$$\begin{aligned} |(T(x_n))_\infty - (T(x))_\infty| &= |\langle M_\infty^u, x_n - x\rangle| \\ &\le |M_\infty^u|_{\Phi^1(U)} \, |x_n - x|_{\Phi[U^0]}. \end{aligned}$$

Since $(x_n - x)$ converges to 0, the sequence is bounded in norm in $\Phi[U^0]$. Hence, by the dominated convergence theorem, $((T(x_n))_\infty)$ converges to $(T(x))_\infty$ in L^2. Hence, $(T(x_n))$ converges to $T(x)$ in M^2 and T is continuous on $\Phi[U^0]$.

Next, suppose B is bounded in Φ. By Proposition 1.5, $B \subseteq U^0$, for some $U \in U_h(\Phi_b^1)$. Then $B \subseteq \Phi[U^0]$ and B is still bounded in $\Phi[U^0]$, since for every x in B, $p_{u^0}(x) \leq 1$.

Since T is continuous on $\Phi[U^0]$ and B is bounded in $\Phi[U^0]$, T(B) is bounded in M^2. Hence, T maps bounded sets into bounded sets which means, since Φ is bornological, that T is continuous on Φ. Since T is a continuous, linear mapping of a nuclear space into a Banach space, T is a nuclear mapping [14, 50.1]. Thus, according to Treves' [14] Proposition 47.2, T has the following representation:

$$(3.3) \qquad T(x) = \sum_{i=1}^{\infty} \lambda_i \langle x_i^1, x \rangle M^i,$$

where $\sum_{i=1}^{\infty} |\lambda_i| < \infty$, (x_i^1) is equicontinuous in Φ^1, and (M^i) is a bounded sequence in M^2.

Then $(x_i^1) \subseteq V^0$ for some $V \in U_h(\Phi)$. We then let

$$(3.4) \qquad N_\infty(w) = \sum_{i=1}^{\infty} \lambda_i x_i^1 M_\infty^i(w).$$

Since $p_{v^0}(x_i^1) \leq 1$ for each i, we have

$$E\Big[\sum_{i=1}^{\infty} |\lambda_i x_i^1 M_\infty^i|_{\Phi^1[V^0]}\Big] \leq E\Big[\sum_{i=1}^{\infty} |\lambda_i||M_\infty^i|\Big]$$

$$= \sum_{i=1}^{\infty} |\lambda_i| E[|M_\infty^i|]$$

$$\leq \sum_{i=1}^{\infty} |\lambda_i| \|M_\infty^i\|_{L^2}$$

$$< \infty,$$

since (M^i) is bounded in M^2 and $\sum_{i=1}^{\infty} |\lambda_i| < \infty$.

Hence, N_∞ is defined (a.s.) and is an integrable $\Phi^1[V^0]$-valued random vector. Moreover, N_∞ is square integrable since

$$\|N_\infty\|_{L^2(\Phi^1[V^0])} \leq \sum_{i=1}^{\infty} \|\lambda_i x_i^1 M_\infty^i\|_{L^2(\Phi^1[V^0])}$$

$$\leq \sum_{i=1}^{\infty} |\lambda_i| \|M_\infty^i\|_{L^2}$$

$$< \infty.$$

We let N^1 be a cadlag version of the $\Phi^1[V^0]$-valued square integrable martingale $E(N_\infty|F_t)$.

In the same manner as N_∞, we may define F_t-measurable, $\Phi^1[V^0]$-valued, square integrable random vectors N_t by

$$N_t = \sum_{i=1}^{\infty} \lambda_i x_i^1 M_t^i.$$

We assert $N_t^1 = N_t$ a.s. For every $[b]$ in $\Phi(V)$, we have, by Proposition 1.3,

$$\langle N_t^1, [b]\rangle = \langle E(N_\infty|F_t), [b]\rangle \text{ a.s.}$$

$$= E(\langle N_\infty, [b]\rangle | F_t)$$

$$= E(\langle N_\infty, b\rangle | F_t)$$

$$= E(\sum_{i=1}^{\infty} \lambda_i \langle x_i^1, b\rangle M_\infty^1 | F_t)$$

$$= E((T(b))_\infty | F_t)$$

$$= (T(b))_t \text{ a.s.}$$

$$= \sum_{i=1}^{\infty} \lambda_i \langle x_i^1, b\rangle M_t^i$$

$$= \langle N_t, b\rangle$$

$$= \langle N_t, [b]\rangle.$$

Letting [b] range through a countable norming set, we obtain $N_t^1 = N_t$ a.s.

We then inject N^1 into Φ^1 and we denote the resulting mapping also by N^1. We now have the required limit.

If $U \in U_h(\Phi_b^1)$, since V^0 is bounded, $V^0 \subseteq \delta U$ for some $\delta > 0$. Thus, $p_u \leq \delta_{p_{V^O}}$. Since N^1 is cadlag in $\Phi[V^0]$, with respect to the norm p_{V^O}, N^1 will also be cadlag in Φ^1, since convergence with respect to p_{V^O} implies convergence with respect to p_u for every U in $U_h(\Phi_b^1)$.

N^1 is weakly adapted, since for every x in Φ, $\langle N_t^1, x\rangle = \langle N_t, x\rangle$ a.s., for every t.

We must lastly show that, for every U in $U_h(\Phi_b^1)$, $k(U)N^1$ and M^u are indistinguishable. Let $x \in \Phi[U^0]$. By Proposition 1.4, we have for each t

$$\begin{aligned} \langle k(U)N_t^1, x\rangle &= \langle N_t^1, x\rangle \\ &= \langle N_t, x\rangle \text{ a.s.} \\ &= (T(x))_t \\ &= \langle M_t^u, x\rangle \text{ a.s.} \end{aligned}$$

Thus, $k(U)N_t^1 = M_t^u$ a.s. in the separable space $\Phi^1(U)$. Hence, $k(U)N^1$ is a modification of M^u. However, N^1 is right continuous and $k(U)$ is continuous, by Proposition 1.2 (b); hence, $k(U)N^1$ is right continuous. Since M^u is also right continuous, we have that $k(U)N^1$ and M^u are indistinguishable.

The proof of Theorem 3 shows that the limit N^1 actually takes values in a separable Hilbert space $\Phi^1[V^0]$, for some $V \in U_h(\Phi)$. It immediately follows that any such limit satisfying Theorem 2 will also take values in $\Phi^1[V^0]$. We state this result as the following:

4 COROLLARY: Let M^1 be a strongly cadlag limit in Φ^1 of a projective system $M = \{M^u \colon U \in U_h(\Phi_b^1)\}$ of square integrable martingale such that, for every U in $U_h(\Phi_b^1)$, $k(U)M^1$ and M^u are indistinguishable. Then, outside of an evanescent set, M^1 takes values in $\Phi^1[V^0]$ for some $V \in U_h(\Phi)$.

Proof: By Theorem 3, we know there exists a strongly cadlag limit in Φ^1 of M, N^1, taking values in $\Phi^1[V^0]$, for some $V \varepsilon U_h(\Phi)$, such that $k(U)N^1$ and M^u are indistinguishable for every U in $U_h(\Phi_b^1)$.

Let $x \varepsilon \Phi$. Then $x \varepsilon \Phi[U^0]$ for some $U \varepsilon U_h(\Phi_b^1)$. Then outside of an evanescent set

$$(4.1) \qquad \begin{aligned} \langle N^1, x\rangle &= \langle k(U)N^1, x\rangle \\ &= \langle M^u, x\rangle \\ &= \langle k(U)M^1, x\rangle \\ &= \langle M^1, x\rangle. \end{aligned}$$

Let $([b_n])$ be a countable basis of the separable Hilbert space $\Phi(V)$, where $b_n \varepsilon \Phi$ is a representative of the equivalence class $[b_n]$. Outside of one evanescent set we have, by (4.1), that

$$\langle M^1, b_n\rangle = \langle N^1, b_n\rangle, \text{ for every } n.$$

Since N^1 takes values in $\Phi^1[V^0]$, the dual of $\Phi(V)$, M^1 may also be defined on $\Phi(V)$ by

$$\langle M^1, [b_n]\rangle = \langle M^1, b_n\rangle, \text{ for each } n.$$

Then, outside of an evanescent set, we have for each n

$$\begin{aligned} \langle M^1, [b_n]\rangle &= \langle N^1, b_n\rangle \\ &= \langle N^1, [b_n]\rangle \end{aligned}$$

and $\langle M^1, \cdot\rangle$ is a continuous, linear mapping on $\Phi(V)$. Hence, outside of an evanescent set, M^1 takes values in the dual of $\Phi(V)$, which is $\Phi^1[V^0]$.

5 REMARKS. Corollary 4 also proves the uniqueness of the limit N^1 in Theorem 3. For suppose that M^1 and N^1 are strongly cadlag limits in Φ^1 of the projective system M such that, for every U in $U_h(\Phi_b^1)$, $k(U)M^1$ and $k(U)N^1$ are indistinguishable from M^u. Then M^1 takes values in $\Phi^1[V_1^0]$ and N^1 takes values in $\Phi^1[V_2^0]$ (up to an evanescent set), for some V_1 and V_2 in $U_h(\Phi)$. We may then choose V in $U_h(\Phi)$ such that V is contained in both V_1 and V_2. Then M^1 and N^1 both take values in $\Phi^1[V^0]$.

As in line (4.1), we obtain for every $x \in \Phi$

$$\langle N^1, x\rangle = \langle M^1, x\rangle, \tag{5.1}$$

outside of an evanescent set. We then let $([b_n])$ be a countable norming set in $\Phi(V)$ for $\Phi^1[V^0]$. Then, outside of one evanescent set, we have

$$\begin{aligned}\langle N^1, [b_n]\rangle &= \langle N^1, b_n\rangle \\ &= \langle M^1, b_n\rangle \\ &= \langle M^1, [b_n]\rangle,\end{aligned}$$

for each n. Hence, N^1 and M^1 are indistinguishable in $\Phi^1[V^0]$, with respect to the norm p_{V^0}.

Then if $U \in U_h(\Phi_b^1)$, since V^0 is bounded in Φ^1, $V^0 \subseteq \delta U$, for some $\delta > 0$. We then have, outside of an evanescent set depending only on p_{v^o}, that

$$p_u(N^1 - M^1) \leq \delta p_{v^o}(N^1 - M^1) = 0.$$

Hence, M^1 and N^1 remain indistinguishable in Φ^1.

We point out that a hypothesis of Theorem 3 is that the projective system M already has a limit, in the sense of Definition 2. The next theorem will eliminate that hypothesis.

6 THEOREM. Let $M = \{M^u: U \in U_h(\Phi_b^1)\}$ be a projective system of square integrable martingales. Then M has a limit in Φ^1.

Proof: We shall use an argument similar to Theorem 3. Let t be fixed. We define a mapping T from Φ into $L^2(\Omega, F_t, P, R)$ by

$$T(x) = \langle M_t^u, x\rangle,$$

where U is any set in $U_h(\Phi_b^1)$ such that $x \in \Phi[U^0]$.

We must verify that T is well-defined. If V is another set in $U_h(\Phi_b^1)$ such that $x \in \Phi[V^0]$, we may choose $W \in U_h(\Phi_b^1)$ such that W is contained in both U and V. Then $x \in \Phi[U^0] \subseteq \Phi[W^0]$ and it suffices to show $\langle M_t^u, x\rangle = \langle M_t^w, x\rangle$. Since the equivalence classes M^u and $k(U, W)M^u$ have the same representative in Φ^1 and $k(U, W)M^u$ is indistinguishable from M^w, we have

$$\langle M_t^u, x\rangle = \langle k(U, W)M_t^u, x\rangle$$

$$= \langle M_t^w, x\rangle \text{ a.s.}$$

Hence, $\langle M_t^u, x\rangle = \langle M_t^w, x\rangle$ in L^2 and T is well-defined.

Similarly, if $x \varepsilon \Phi[U^0]$ and $y \varepsilon \Phi[V^0]$, we choose W contained in both U and V; then for every scalar a, $ax + y \varepsilon \Phi[W^0]$ and

$$T(ax + y) = \langle M_t^w, ax+y\rangle$$

$$= a\langle M_t^w, x\rangle + \langle M_t^w, y\rangle$$

$$= a\langle M_t^u, x\rangle + \langle M_t^u, y\rangle$$

$$= aT(x) + T(y).$$

Hence, T is linear. To show T is continuous, it suffices, as in Theorem 3, to show T is continuous on $\Phi[U^0]$, for every $U \varepsilon U_h(\Phi_b^1)$.

If (x_n) converges to x in $\Phi[U^0]$, then

$$|T(x_n) - T(x)| = |\langle M_t^u, x_n-x\rangle|$$

$$\leq |M_t^u|_{\Phi^1(U)} |x_n-x|_{\Phi[U^0]}.$$

Hence, by the dominated convergence theorem, $(T(x_n))$ converges to T(x) in L^2. T is then continuous on $\Phi[U^0]$ and hence on all of Φ. T is then a nuclear mapping and has the following representation:

$$T(x) = \sum_{i=1}^{\infty} \lambda_i \langle x_i^1, x\rangle M_t^i,$$

where $\sum_{i=1}^{\infty} |\lambda_i| < \infty$, (x_i^1) is equicontinuous in Φ^1, and (M_t^i) is bounded in $L^2(\Omega, F_t, P, R)$.

We choose $V \in U_h(\Phi)$ such that $(x_i^1) \subseteq V^0$. We then let

$$M_t^1 = \sum_{i=1}^{\infty} \lambda_i x_i^1 M_t^i.$$

Since $p_{V^0}(x_i^1) \leq 1$ for each i, we have

$$E[\sum_{i=1}^{\infty} |\lambda_i x_i^1 M_t^i|_{\Phi^1[V^0]}] \leq \sum_{i=1}^{\infty} |\lambda_i| E[|M_t^i|]$$

$$< \infty.$$

Hence, the sum M_t^1 is absolutely convergent, and thus defined, almost surely in $\Phi[V^0]$. We then inject M_t^1 into Φ^1 and we denote the resulting mapping again by M_t^1.

M_t^1 is then weakly F_t-measurable, since for every $x \in \Phi$, $\langle M_t^1, x\rangle = T(x)$ a.s. Lastly, we must show that $k(U)M_t^1 = M_t^u$ a.s., for every $U \in U_h(\Phi_b^1)$. If $x \in \Phi[U^0]$, then

$$\begin{aligned} \langle k(U)M_t^1, x\rangle &= \langle M_t^1, x\rangle \\ &= T(x) \quad \text{a.s.} \\ &= \langle M_t^u, x\rangle. \end{aligned}$$

Hence, $k(U)M_t^1$ and M_t^u remain equal a.s. in the separable space $\Phi^1(U)$.

Combining Theorem 3 and 6 and Remark 5, we have

7 THEOREM. Every projective system $M = \{M^u: U \in U_h(\Phi_b^1)\}$ of square integrable martingales has a unique strongly cadlag limit M^1 in Φ^1 such that, for every $U \in U_h(\Phi_b^1)$, $k(U)M^1$ is indistinguishable from M^u.

In the case of Theorem 7, we often denote both the projective system and its limit by M.

We conclude this section with some additional definitions.

8 DEFINITION. Let $X = \{X^u: U \in U_h(\Phi_b^1)\}$ and $Y = \{Y^u: U \in U_h(\Phi_b^1)\}$ be projective systems of processes. We say X and Y are indistinguishable (resp. equivalent) if X^u is indistinguishable from (resp. is a modification of) Y^u, for every $U \in U_h(\Phi_b^1)$.

9 DEFINITION. Let $X = \{X^u: U \in U_h(\Phi_b^1)\}$ and $Y = \{Y^u: U \in U_h(\Phi_b^1)\}$ be projective systems of processes and let T be a stopping time.

(a) We define the set X + Y by $X + Y = \{X^u + Y^u: U \in U_h(\Phi_b^1)\}$.

(b) We define the set X^T by $X^T = \{(X^u)^T: U \in U_h(\Phi_b^1)\}$.

It is obvious that X + Y and X^T are projective systems.

10 LEMMA. If $X = \{X^u: U \in U_h(\Phi_b^1)\}$ is a projective system of cadlag processes, then $\Delta X = \{\Delta X^u: U \in U_h(\Phi_b^1)\}$ is a projective system.

Proof: If U and V are in $U_h(\Phi_b^1)$ with $V \subseteq U$, we have, outside of an evanescent set, that $k(U, V)X^v = X^u$. Since $k(U, V)$ is continuous and linear, we have $k(U, V)\Delta X^v = \Delta X^u$.

3. Projective Systems of Optional Integrals

We now naturally extend the optional integrals defined in Chapters III and IV to projective systems of semimartingales. Many properties of the resulting projective systems will be obvious, since they reduce to properties of the optional integral in Hilbert space.

For every U in $U_h(\Phi_b^1)$, we shall identify the separable Hilbert spaces $\Phi^1(U)$ and $\Phi[U^0]$, which are isometrically isomorphic by Proposition 1.4. We shall continue to use $<\cdot, \cdot>$ to represent the inner product in these spaces.

1 THEOREM. Let $X = \{X^u: U \in U_h(\Phi_b^1)\}$ be a projective system of special semimartingales. Let H be a real optional process such that $H \in L_0(X^u)$ for every $U \in U_h(\Phi_b^1)$. Then the set $H \underset{c}{\cdot} X = \{H \underset{c}{\cdot} X^u: U \in U_h(\Phi_b^1)\}$ is a projective system of semimartingales.

$H \underset{c}{\cdot} X$ is called the optional stochastic integral of H with respect to X and we denote $(H \underset{c}{\cdot} X)^u = H \underset{c}{\cdot} X^u$, for every $U \in U_h(\Phi_b^1)$.

If H is locally bounded and predictable, we write $H \cdot X = \{H \cdot X^u: U \in U_h(\Phi_b^1)\}$ and we may take X to be a projective system of semimartingales.

Proof: Let V and U be in $U_h(\Phi_b^1)$ with $V \subseteq U$. Let $x \in \Phi[U^0] \subseteq \Phi[V^0]$. Since the equivalence classes $k(U, V)H \underset{c}{\cdot} X^v$ and $H \underset{c}{\cdot} X^v$ have the same representative in Φ^1, the equivalence classes X^V and $k(U, V)X^V$ have the same representative in Φ^1, and $k(U, V)X^V$ is indistinguishable from X^u, we have, by Corollary III.3.4,

$$\begin{aligned} \langle x, k(U, V)H\underset{c}{\cdot}X^V\rangle &= \langle x, H\underset{c}{\cdot}X^V\rangle \\ &= H\underset{c}{\cdot}\langle x, X^V\rangle \\ &= H\underset{c}{\cdot}\langle x, k(U, V)X^V\rangle \\ &= H\underset{c}{\cdot}\langle x, X^u\rangle \\ &= \langle x, H\underset{c}{\cdot}X^u\rangle, \end{aligned}$$

outside of an evanescent set. Hence, in the separable space $\Phi^1(U)$, $k(U, V)H\underset{c}{\cdot}X^V$ and $H\underset{c}{\cdot}X^u$ are indistinguishable and $H\underset{c}{\cdot}X$ is a projective system of semimartingales.

If H is locally bounded and predictable and X is a projective system of semimartingales, we may use the same proof, by replacing optional integrals with predictable integrals, using Theorem II.2.6.

2 REMARKS.

(a) If H is locally bounded and predictable and X is a projective system of special semimartingales, then $H \cdot X$ is a projective system of special semimartingales.

(b) If X is a projective system of square integrable martingales and $H \in L_0^2(X^u)$, for every $U \in U_h(\Phi_b^1)$, then $H\underset{c}{\cdot}X$ is also a projective system of square integrable martingales. Hence, by Theorem 2.7, $H\underset{c}{\cdot}X$ has a unique limit in Φ^1.

(c) Suppose, for every $U \in \mathcal{U}_h(\Phi_b^1)$, X^u has canonical decomposition $X^u = X_0^u + M^u + A^u$, where M^u is a local martingale of integrable variation, and A^u is a predictable process of integrable variation, and suppose H is bounded. Then

$$H \underset{c}{\cdot} X^u = H_0 X_0^u + H \cdot M^u - (H \cdot M^u)^p + H \cdot A^u$$

is also of integrable variation. Hence, $H \underset{c}{\cdot} X$ again has a limit in Φ^1 [15, II.2].

An immediate consequence of Definitions 2.8 and 2.9 is the following result.

3 THEOREM. Let $X = \{X^u : U \in \mathcal{U}_h(\Phi_b^1)\}$ be a projective system of special semimartingales and let H be a real, optional process such that $H \in L_0(X^u)$ for each $U \in \mathcal{U}_h(\Phi_b^1)$.

(a) If K is another real, optional process such that $K \in L_0(X^u)$ for each $U \in \mathcal{U}_h(\Phi_b^1)$, then $(H + K) \underset{c}{\cdot} X$ and $H \underset{c}{\cdot} X + K \underset{c}{\cdot} X$ are indistinguishable.

(b) If Y is a projective system of special semimartingales and $H \in L_0(Y^u)$ for each $U \in \mathcal{U}_h(\Phi_b^1)$, then $H \underset{c}{\cdot} (X + Y)$ and $H \underset{c}{\cdot} X + H \underset{c}{\cdot} Y$ are indistinguishable.

(c) If T is a stopping time, then the projective systems $(H \underset{c}{\cdot} X)^T$, $H 1_{[0,T]} \underset{c}{\cdot} X$, and $H \underset{c}{\cdot} X^T$ are indistinguishable.

Proof: By Theorems III.4.1 and III.4.2, we have for every $U \in U_h(\Phi_b^1)$, up to indistinguishability,

(a) $$((H + K) \underset{c}{\bullet} X)^u = (H + K) \underset{c}{\bullet} X^u$$

$$= H \underset{c}{\bullet} X^u + K \underset{c}{\bullet} X^u$$

$$= (H \underset{c}{\bullet} X)^u + (K \underset{c}{\bullet} X)^u$$

$$= (H \underset{c}{\bullet} X + K \underset{c}{\bullet} X)^u.$$

(b) $$(H \underset{c}{\bullet} (X + Y))^u = H \underset{c}{\bullet} (X^u + Y^u)$$

$$= H \underset{c}{\bullet} X^u + H \underset{c}{\bullet} Y^u$$

$$= (H \underset{c}{\bullet} X)^u + (H \underset{c}{\bullet} Y)^u$$

$$= (H \underset{c}{\bullet} X + H \underset{c}{\bullet} Y)^u.$$

(c) $$((H \underset{c}{\bullet} X)^T)^u = (H \underset{c}{\bullet} X^u)^T$$

$$= H1_{[0,T]} \underset{c}{\bullet} X^u$$

$$= (H1_{[0,T]} \underset{c}{\bullet} X)^u,$$

and

$$((H \underset{c}{\bullet} X)^T)^u = (H \underset{c}{\bullet} X^u)^T$$

$$= H \underset{c}{\bullet} (X^u)^T$$

$$= (H \underset{c}{\bullet} X^T)^u.$$

We now give the decomposition theorem for a projective system of local martingales.

4 THEOREM. Let $X = \{X^u : U \in U_h(\Phi_b^1)\}$ be a projective system of local martingales. Then the sets $X^d = \{(X^u)^d : U \in U_h(\Phi_b^1)\}$ and $X^c = \{(X^u)^c : U \in U_h(\Phi_b^1)\}$ are projective systems and $X = X^c + X^d$.

Proof: It is clear that $X = X^c + X^d$. Let V and U be in $U_h(\Phi_b^1)$ with $V \subset U$.

We let $J = 1_{\{p_u(\Delta X^u) \neq 0\}}$ and $H = 1_{\{p_v(\Delta X^v) \neq 0\}}$. By Theorem III.5.5, $(X^v)^d = H \underset{c}{\cdot} X^v$. Since H is bounded and optional, we have, by Theorem 1, $k(U, V)(X^v)^d = k(U, V) H \underset{c}{\cdot} X^v = H \underset{c}{\cdot} X^u$. Thus, we must show $H \underset{c}{\cdot} X^u = (X^u)^d$. Since $(X^u)^d = J \underset{c}{\cdot} X^u$, it suffices to show $H \underset{c}{\cdot} X^u - J \underset{c}{\cdot} X^u = (H - J) \underset{c}{\cdot} X^u = 0$. We let $L = H - J$. Since X^u and X^v are cadlag, L is thin. Moreover, since X is a projective system, $k(U, V)\Delta X^v$ is indistinguishable from ΔX^u.

Then for every bounded $\Phi^1(U)$-valued martingale Y

$$\int^t L_s \, d[X^u, Y]_s = \sum_{s \leq t} L_s \Delta[X^u, Y]_s$$

$$= \sum_{s \leq t} L_s \langle \Delta X_s^u, \Delta Y_s \rangle$$

$$= \sum_{s \leq t} [1_{\{p_v(\Delta X_s^v) \neq 0\}} - 1_{\{p_u(k(U, V)\Delta X_s^v) \neq 0\}}] \langle \Delta X_s^u, \Delta Y_s \rangle .$$

For each s, if $p_u(\Delta X_s^u) = 0$, then the summand is 0, since $\langle \Delta X_s^u, \Delta Y_s \rangle = 0$.

Suppose $p_u(\Delta X_s^u) \neq 0$. Then, since $V \subseteq U$, we have $p_u \leq p_v$ and

$$\begin{aligned} 0 &< p_u(\Delta X_s^u) \\ &= p_u(k(U, V)\Delta X_s^v) \\ &\leq p_v(k(U, V)\Delta X_s^v) \\ &= p_v(\Delta X_s^v); \end{aligned}$$

hence,

$$1_{\{p_v(\Delta X_s^v)\neq 0\}} - 1_{\{p_u(k(U, V)\Delta X_s^v)\neq 0\}} = 0.$$

Thus, $\int^t L_s d[X^u, Y]_s = 0$ and $[0, Y] - L\cdot[X^u, Y] = 0$ is a local martingale which is zero at 0. By Theorem III.3.3, $L \underset{c}{\cdot} X^u = 0$. Hence, $k(U, V)(X^v)^d = (X^u)^d$ and X^d is a projective system. Since

$$\begin{aligned} k(U, V)(X^v)^c &= k(U, V)(X^v - (X^v)^d) \\ &= X^u - (X^u)^d \\ &= (X^u)^c, \end{aligned}$$

X^c is also a projective system.

In the same manner as Theorem 1, we shall now extend the optional integral of Chapter IV to a projective system.

5 THEOREM. Let $H = \{H^u\colon U \in U_h(\Phi_b^1)\}$ be a projective system of locally bounded, optional processes and let X be a real special semimartingale. Then the set $H \underset{c}{\cdot} X = \{H^u \underset{c}{\cdot} X\colon U \in U_h(\Phi_b^1)\}$ is a projective system of semimartingales.

$H \underset{c}{\cdot} X$ is called the optional stochastic integral of H with respect to X and we denote $(H \underset{c}{\cdot} X)^u = H^u \underset{c}{\cdot} X$, for every $U \in U_h(\Phi_b^1)$.

If H is a projective system of locally bounded, predictable processes, we write $H \cdot X = \{H^u \cdot X \colon U \in U_h(\Phi_b^1)\}$ and we may take X to be an arbitrary semimartingale.

Proof: Let V and U be in $U_h(\Phi_b^1)$ with $V \subseteq U$. Let $x \in \Phi[U^0] \subseteq \Phi[V^0]$. Since the equivalence classes $k(U, V)(H^v \underset{c}{\cdot} X)$ and $H^v \underset{c}{\cdot} X$ have the same representative in Φ^1, the equivalence classes H^v and $k(U, V)H^v$ have the same representative in Φ^1, and $k(U, V)$ is indistinguishable from H^u, we have, by Corollary IV.3.2,

$$\begin{aligned} \langle x, k(U, V)(H^v \underset{c}{\cdot} X)\rangle &= \langle x, H^v \underset{c}{\cdot} X\rangle \\ &= \langle x, H^v\rangle \underset{c}{\cdot} X \\ &= \langle x, k(U, V)H^v\rangle \underset{c}{\cdot} X \\ &= \langle x, H^u\rangle \underset{c}{\cdot} X \\ &= \langle x, H^u \underset{c}{\cdot} X\rangle, \end{aligned}$$

outside of an evanescent set. Hence, in the separable space $\Phi^1(U)$, $k(U, V)H^v \underset{c}{\cdot} X$ and $H^u \underset{c}{\cdot} X$ are indistinguishable and $H \underset{c}{\cdot} X$ is a projective system of semimartingales.

If H is a projective system of locally bounded, predictable processes and X is a semimartingale, we may use the same proof, by replacing optional integrals with predictable integrals, using Theorem II.2.6.

6 REMARKS.

(a) If H is a projective system of locally bounded, predictable processes and X is a special semimartingale, then $H \cdot X$ is a projective system of special semimartingales.

(b) If X is a square integrable martingale and $H^u \in L_0^2(X)$, for each $U \in U_h(\Phi_b^1)$, then $H \underset{c}{\cdot} X$ is a projective system of square integrable martingales, for which there is a unique limit in Φ^1, by Theorem 2.7.

(c) If X has canonical decomposition $X = X_0 + M + A$, where M is a local martingale of integrable variation, and A is a predictable process of integrable variation, and H^u is bounded for each $U \in U_h(\Phi_b^1)$, then

$$H^u \underset{c}{\cdot} X = H_0^u X_0 + H^u \cdot M - (H^u \cdot M)^p + H^u \cdot A$$

is also of integrable variation. Hence, $H \underset{c}{\cdot} X$ again has a limit in Φ^1 [15, II.2].

We may immediately obtain a result analogous to Theorem 3.

7 THEOREM. Let $H = \{H^u\colon U \in U_h(\Phi_b^1)\}$ be a projective system of locally bounded, optional processes and let X be a real special semimartingale.

(a) If $K = \{K^u\colon U \in U_h(\Phi_b^1)\}$ is a projective system of locally bounded, optional processes, then $(H + K) \underset{c}{\cdot} X$ and $H \underset{c}{\cdot} X + K \underset{c}{\cdot} X$ are indistinguishable.

(b) If Y is a real special semimartingale, then $H \underset{c}{\cdot} (X + Y)$ and $H \underset{c}{\cdot} X + H \underset{c}{\cdot} Y$ are indistinguishable.

(c) If T is a stopping time, then $H1_{[0,T]} = \{H^u 1_{[0,T]} : U \in U_h(\Phi_b^1)\}$ is a projective system and $H1_{[0,T]} \underset{c}{\cdot} X$, $H \underset{c}{\cdot} X^T$, and $(H \underset{c}{\cdot} X)^T$ are all indistinguishable.

Proof: By Theorems IV.4.1 and IV.4.2, we have for every $U \in U_h(\Phi_b^1)$, up to indistinguishability,

(a)
$$\begin{aligned} ((H + K) \underset{c}{\cdot} X)^u &= (H^u + K^u) \underset{c}{\cdot} X \\ &= H^u \underset{c}{\cdot} X + K^u \underset{c}{\cdot} X \\ &= (H \underset{c}{\cdot} X)^u + (K \underset{c}{\cdot} X)^u \\ &= (H \underset{c}{\cdot} X + K \underset{c}{\cdot} X)^u. \end{aligned}$$

(b)
$$\begin{aligned} (H \underset{c}{\cdot} (X + Y))^u &= H^u \underset{c}{\cdot} (X + Y) \\ &= H^u \underset{c}{\cdot} X + H^u \underset{c}{\cdot} Y \\ &= (H \underset{c}{\cdot} X)^u + (H \underset{c}{\cdot} Y)^u \\ &= (H \underset{c}{\cdot} X + H \underset{c}{\cdot} Y)^u. \end{aligned}$$

(c) If $V \in U_h(\Phi_b^1)$ and $V \subseteq U$, then $k(U, V)(H^v 1_{[0,T]}) = (k(U, V)H^v)1_{[0,T]} = H^u 1_{[0,T]}$. Moreover,

$$((H \underset{c}{\cdot} X)^T)^u = (H^u \underset{c}{\cdot} X)^T$$

$$= H^u 1_{[0,T]} \underset{c}{\bullet} X$$

$$= (H 1_{[0,T]} \underset{c}{\bullet} X)^u,$$

and

$$((H \underset{c}{\bullet} X)^T)^u = (H^u \underset{c}{\bullet} X)^T$$

$$= H^u \underset{c}{\bullet} X^T$$

$$= (H \underset{c}{\bullet} X^T)^u.$$

8 REMARKS. Theorems 1 and 5 simply define projective systems. If desired, we may characterize these systems in the following manner: In the case of Theorem 1,

(a) $H \underset{c}{\bullet} X$ is the unique (up to indistinguishability) projective system such that, for every $U \in U_h(\Phi_b^1)$,

$$[(H \underset{c}{\bullet} X)^u, Y] - H \bullet [X^u, Y]$$

is a local martingale which is zero at 0, for every bounded, $\Phi^1(U)$-valued martingale Y.

In the case of Theorem 5,

(b) $H \underset{c}{\bullet} X$ is the unique (up to indistinguishability) projective system such that, for every $U \in U_h(\Phi_b^1)$,

$$(H \underset{c}{\bullet} X)^u Y - H^u \bullet [X, Y]$$

is a local martingale which is zero at 0, for every real bounded martingale Y.

Using the associativity properties of Theorem III.4.4 and IV.4.4, we immediately obtain the following result.

9 THEOREM.

(a) Let $X = \{X^u : U \in U_h(\Phi_b^1)\}$ be a projective system of special semimartingales. Let H be a real, locally bounded, optional process and let K be a real, locally bounded, predictable process. Then $HK \underset{c}{\cdot} X = H \underset{c}{\cdot} (K \cdot X) = K \cdot (H \underset{c}{\cdot} X)$, as indistinguishable projective systems.

(b) Let $H = \{H^u : U \in U_h(\Phi_b^1)\}$ be a projective system of locally bounded, optional processes. Let X be a real special semimartingale and let K be a real, locally bounded, predictable process. Then $HK \underset{c}{\cdot} X = H \underset{c}{\cdot} (K \cdot X) = K \cdot (H \underset{c}{\cdot} X)$, as projective systems.

(c) Let $K = \{K^u : U \in U_h(\Phi_b^1)\}$ be a projective system of locally bounded, predictable processes. Let X be a real special semimartingale and let H be a real, locally bounded, optional process. Then $HK \underset{c}{\cdot} X = H \underset{c}{\cdot} (K \cdot X) = K \cdot (H \underset{c}{\cdot} X)$, as projective systems.

Proof:

(a) For every $U \in U_h(\Phi_b^1)$,

$$\begin{aligned}(HK \underset{c}{\cdot} X)^u &= HK \underset{c}{\cdot} X^u \\ &= H \underset{c}{\cdot} (K \cdot X^u)\end{aligned}$$

$$\begin{aligned} &= H \underset{c}{\cdot} (K \cdot X)^{u} \\ &= (H \underset{c}{\cdot} (K \cdot X))^{u}, \end{aligned}$$

and

$$\begin{aligned} (HK \underset{c}{\cdot} X)^{u} &= HK \underset{c}{\cdot} X^{u} \\ &= K \cdot (H \underset{c}{\cdot} X^{u}) \\ &= K \cdot (H \underset{c}{\cdot} X)^{u} \\ &= (K \cdot (H \underset{c}{\cdot} X))^{u}. \end{aligned}$$

(b) We first note that $HK = \{H^{u}K\colon U \in U_{h}(\Phi_{b}^{1})\}$ is a projective system of locally bounded, optional processes. Then for every $U \in U_{h}(\Phi_{b}^{1})$,

$$\begin{aligned} (HK \underset{c}{\cdot} X)^{u} &= H^{u}K \underset{c}{\cdot} X \\ &= H^{u} \underset{c}{\cdot} (K \cdot X) \\ &= (H \underset{c}{\cdot} (K \cdot X))^{u}, \end{aligned}$$

and

$$\begin{aligned} (HK \underset{c}{\cdot} X)^{u} &= H^{u}K \underset{c}{\cdot} X \\ &= K \cdot (H^{u} \underset{c}{\cdot} X) \\ &= K \cdot (H \underset{c}{\cdot} X)^{u} \\ &= (K \cdot (H \underset{c}{\cdot} X))^{u}. \end{aligned}$$

The proof of (c) is similar to (b).

Let M be a projective system of square integrable martingales. By Theorem 2.7, M has a unique limit in Φ^1, which we also denote by M. M is then simply called a square integrable martingale in Φ^1. We let $M(\Phi^1)$ be the set of square integrable martingales in Φ^1. Ustanel [15] showed that $M(\Phi^1)$ is a complete, reflexive, locally convex space under the topology induced by the seminorms

$$g_u(M) = (E[\|M_\infty^u\|^2])^{1/2}, \; U \in U_h(\Phi_b^1),$$

where $\|M_\infty^u\|$ means the norm of M_∞^u in $\Phi^1(u)$.

We may then immediately obtain convergence theorems for the optional integrals in $M(\Phi^1)$.

10 THEOREM.

(a) Let M be a square integrable martingale in Φ^1. Let (H^k) be a sequence of real, uniformly bounded, optional processes which converge pointwise to H. Then $(H^k {}_c\!\cdot X)$ converges to $H {}_c\!\cdot X$ in $M(\Phi^1)$.

(b) Let H be a real, bounded, optional process. Let (M^k) be a sequence of square integrable martingales converging to M in $M(\Phi^1)$. Then $(H {}_c\!\cdot M^k)$ converges to $H {}_c\!\cdot M$ in $M(\Phi^1)$.

Proof:

(a) For every $U \in U_h(\Phi_b^1)$,

$$g_u(H^k {}_c\!\cdot M - H {}_c\!\cdot M) = g_u((H^k - H) {}_c\!\cdot M)$$

$$= \|((H^k - H)\underset{c}{\cdot}M)^u\|_{M^2_{\Phi^1}(u)}$$

$$= \|(H^k - H)\underset{c}{\cdot}M^u\|_{M^2_{\Phi^1}(u)} \quad .$$

Thus, $\lim_k g_u(H^k\underset{c}{\cdot}M - H\underset{c}{\cdot}M) = 0$, by Theorem III.1.6 (a).

(b) The result follows from Theorem III.1.6 (b), since for every $U \in U_h(\Phi^1_b)$,

$$g_u(H\underset{c}{\cdot}M^k - H\underset{c}{\cdot}M) = g_u(H\underset{c}{\cdot}(M^k - M))$$

$$= \|(H\underset{c}{\cdot}(M^k - M))^u\|_{M^2_{\Phi^1}(u)}$$

$$= \|H\underset{c}{\cdot}((M^k)^u - M^u)\|_{M^2_{\Phi^1}(u)}$$

$$= \|H\underset{c}{\cdot}(M^k)^u - H\underset{c}{\cdot}M^u\|_{M^2_{\Phi^1}(u)} \quad .$$

11 THEOREM.

(a) Let M be a real square integrable martingale. Let (H^k) be a sequence of projective systems of bounded, optional processes. Let H be a projective system of bounded, optional processes. Suppose for every $U \in U_h(\Phi^1_b)$, $((H^k)^u)$ is uniformly bounded in k and converges pointwise to H^u. Then $(H^k\underset{c}{\cdot}M)$ converges in $M(\Phi^1)$ to $H\underset{c}{\cdot}M$.

(b) Let H be a projective system of bounded, optional processes. Let (M^k) converge to M in M^2. Then $(H\underset{c}{\cdot}M^k)$ converges to $H\underset{c}{\cdot}M$ in $M(\Phi^1)$.

Proof:

(a) For every $U \in U_h(\Phi_b^1)$,

$$g_u(H^k\underset{c}{\cdot}M - H\underset{c}{\cdot}M) = \|(H^k)^u\underset{c}{\cdot}M - H^u\underset{c}{\cdot}M\|_{M^2_{\Phi^1(u)}} .$$

The result follows by Theorem IV.1.6 (a).

(b) For every $U \in U_h(\Phi_b^1)$,

$$g_u(H\underset{c}{\cdot}M^k - H\underset{c}{\cdot}M) = \|H^u\underset{c}{\cdot}M^k - H^u\underset{c}{\cdot}M\|_{M^2_{\Phi^1(u)}} .$$

The result follows by Theorem IV.1.6 (b).

4. The Optional Integral in Nuclear Space

Using the duality of Φ and Φ^1, we shall now extend the optional integral of Chapter V to integrate a bounded, (weakly) optional, Φ-valued process with respect to a projective system of special semimartingales. As in Section 3, we shall continue to identify the spaces $\Phi^1(U)$ and $\Phi[U^0]$, for every $U \in U_h(\Phi_b^1)$, and continue to use $\langle\cdot, \cdot\rangle$ to represent the inner product.

1 LEMMA. Let $X = \{X^u: U \in U_h(\Phi_b^1)\}$ be a projective system of local martingales which are zero at 0. Let V and U be in $U_h(\Phi_b^1)$ such that $V \subseteq U$. If Z is a $\Phi[U^0]$-valued local martingale, then

$$[Z, X^u] = [Z, X^v].$$

Proof: We note that $\Phi[U^0] \subseteq \Phi[V^0]$; thus, $[Z, X^v]$ is defined. Since X^u and $k(U, V)X^v$ are indistinguishable and $k(U, V)X^v$ has the same representative in Φ^1 as the equivalence class X^v, we have

$$\begin{aligned} \langle Z, X^u\rangle - [Z, X^v] &= \langle Z, k(U, V)X^v\rangle - [Z, X^v] \\ &= \langle Z, X^v\rangle - [Z, X^v] \end{aligned}$$

is a local martingale which is zero at 0. Moreover,

$$\begin{aligned} \Delta[Z, X^v] &= \langle \Delta Z, \Delta X^v\rangle \\ &= \langle \Delta Z, k(U, V)\Delta X^v\rangle \\ &= \langle \Delta Z, \Delta X^u\rangle. \end{aligned}$$

Hence, $[Z, X^v] = [Z, X^u]$, by Theorem II.3.5.

2 LEMMA. Let $A = \{A^u: U \in U_h(\Phi_b^1)\}$ be a projective system of processes of finite variation. Let V and U be in $U_h(\Phi_b^1)$ such that $V \subseteq U$. If H is a bounded, $\Phi[U^0]$-valued, measurable process, then $H \cdot A^u = H \cdot A^v$.

Proof: Since H takes values in $\Phi[U^0] \subseteq \Phi[V^0]$, and A^u is indistinguishable from $k(U, V)A^v$, we have, outside of an evanescent set,

$$\langle H_s, A^u_t\rangle = \langle H_s, k(U, V)A^v_t\rangle$$
$$= \langle H_s, A^v_t\rangle,$$

for all s and t.

Then if $\{0 = t_0, t_1, t_2, \ldots, t_n = t\}$ is a partition of $[0, t]$ and $s_i \in]t_i, t_{i+1}]$, for $i = 0, 1, \ldots, n$, we have

$$\langle H_0, A^u_0\rangle + \sum_{i=0}^{n} \langle H_{s_i}, A^u_{t_{i+1}} - A^u_{t_i}\rangle = \langle H_0, A^v_0\rangle + \sum_{i=0}^{n} \langle H_{s_i}, A^v_{t_{i+1}} - A^v_{t_i}\rangle.$$

It follows that $\int^t H_s dA^u_s = \int^t H_s dA^v_s$ and $H\cdot A^u$ and $H\cdot A^v$ are indistinguishable.

We now come to the definition of the optional integral in nuclear space.

Suppose H is a weakly optional, Φ-valued mapping on $R^+ \times \Omega$ having a bounded range. By Proposition 1.5, H takes values and is bounded in a separable Hilbert space $\Phi[U^0]$, for some $U \in U_h(\Phi^1_b)$. Moreover, for every $[b] \in \Phi^1(U)$, where b is a representative in Φ^1 of $[b]$, we have $\langle H, [b]\rangle = \langle H, b\rangle$ is optional. H is then weakly optional, and hence optional in $\Phi[U^0]$. If X^u is a $\Phi^1(U)$- valued special semimartingale, we may then define $H \underset{c}{\cdot} X^u$, according to Theorem V.2.2. We may, therefore, make the following definition.

3 DEFINITION. Let H be a weakly optional Φ-valued mapping on $R^+ \times \Omega$ having a bounded range and let $X = \{X^u : U \in U_h(\Phi_b^1)\}$ be a projective system of special semimartingales. We define the optional stochastic integral of H with respect to X to be the real semimartingale $H \underset{c}{\cdot} X^u$, where $U \in U_h(\Phi_b^1)$ absorbs the range of H. We denote this process by $H \underset{c}{\cdot} X$. If H is weakly predictable, we write $H \cdot X$ and we may take X to be a projective system of semimartingales.

4 THEOREM. $H \underset{c}{\cdot} X$ is a well-defined process, independent of the choice of $U \in U_h(\Phi_b^1)$ absorbing the range of H.

Proof: Suppose U and W are in $U_h(\Phi_b^1)$ such that $\Phi[U^0]$ and $\Phi[W^0]$ contain the range of H. We then choose $V \in U_h(\Phi_b^1)$ such that V is contained in both U and W. Then $\Phi[U^0] \subseteq \Phi[V^0]$ and $\Phi[W^0] \subseteq \Phi[V^0]$. It then suffices to show $H \underset{c}{\cdot} X^u = H \underset{c}{\cdot} X^v$.

Since X is a projective system of special semimartingales, there exists two projective systems [15, III.1] $M = \{M^u : U \in U_h(\Phi_b^1)\}$ and $A = \{A^u : U \in U_h(\Phi_b^1)\}$ such that, for every $U \in U_h(\Phi_b^1)$, M^u is a local martingale, A^u is a predictable process of finite variation, $M_0^u = 0 = A_0^u$, and $X^u = X_0^u + M^u + A^u$. Then $H \underset{c}{\cdot} X^u = \langle H_0, X_0^u \rangle + H \underset{c}{\cdot} M^u + H \cdot A^u$ and $H \underset{c}{\cdot} X^v = \langle H_0, X_0^v \rangle + H \underset{c}{\cdot} M^v + H \cdot A^v$.

Since $k(U, V)(X^v - M^v - A^v)$ is indistinguishable from $X^u - M^u - A^u$, we have $k(U, V)X_0^v = X_0^u$ a.s. Hence,

$$\langle H_0, X_0^u \rangle = \langle H_0, k(U, V)X_0^v \rangle \text{ a.s.}$$

$$= \langle H_0, X_0^v \rangle .$$

By Lemma 2, we have $H \cdot A^u = H \cdot A^v$.

To show $H \underset{c}{\cdot} M^u = H \underset{c}{\cdot} M^v$, it suffices to show $[H \underset{c}{\cdot} M^u, Y] = [H \underset{c}{\cdot} M^v, Y]$ for every real bounded martingale Y (by the proof of uniqueness in Theorem V.2.2).

We let H^1 be a bounded, $\Phi[U^0]$-valued, predictable process such that $H - H^1$ is thin and $H_0^1 = H_0$. We then decompose $[H \underset{c}{\cdot} M^u, Y]$ as $[H \underset{c}{\cdot} M^u, Y] = [H \underset{c}{\cdot} M^u, Y]^c + [H \underset{c}{\cdot} M^u, Y]^d$. We then have

$$\begin{aligned}
[H \underset{c}{\cdot} M^u, Y]^c &= [(H \underset{c}{\cdot} M^u)^c, Y^c] \\
&= [H \underset{c}{\cdot} (M^u)^c, Y^c] \\
&= [H^1 \cdot (M^u)^c, Y^c] \\
&= [(M^u)^c, H^1 \cdot Y^c] \\
&= [(M^v)^c, H^1 \cdot Y^c] \\
&= [H^1 \cdot (M^v)^c, Y^c] \\
&= [H \underset{c}{\cdot} (M^v)^c, Y^c] \\
&= [(H \underset{c}{\cdot} M^v)^c, Y^c] \\
&= [H \underset{c}{\cdot} M^v, Y]^c ,
\end{aligned}$$

by Theorems III.5.9, V.3.6, V.3.7, II.4.8, and for the fifth equality Theorem 3.4 and Lemma 1.

Since

$$\begin{aligned}\langle H, \Delta M^u\rangle &= \langle H, k(U, V)\Delta M^v\rangle \\ &= \langle H, \Delta M^v\rangle,\end{aligned}$$

we have, by Theorem V.3.3,

$$\begin{aligned}\Delta(H_c\cdot M^u) &= \langle H, \Delta M^u\rangle - {}^p\langle H, \Delta M^u\rangle \\ &= \langle H, \Delta M^v\rangle - {}^p\langle H, \Delta M^v\rangle \\ &= \Delta(H_c\cdot M^v>.\end{aligned}$$

Hence,

$$\begin{aligned}[H_c\cdot M^u, Y]^d_t &= \sum_{s\le t} \Delta[H_c\cdot M^u, Y]_s \\ &= \sum_{s\le t} \Delta(H_c\cdot M^u)_s \Delta Y_s \\ &= \sum_{s\le t} \Delta(H_c\cdot M^v)_s \Delta Y_s \\ &= \sum_{s\le t} \Delta[H_c\cdot M^v, Y]_s \\ &\ [H_c\cdot M^v, Y]^d_t.\end{aligned}$$

Thus,

$$\begin{aligned}[H_c\cdot M^u, Y] &= [H_c\cdot M^u, Y]^c + [H_c\cdot M^u, Y]^d \\ &= [H_c\cdot M^v, Y]^c + [H_c\cdot M^v, Y]^d \\ &= [H_c\cdot M^v, Y]\end{aligned}$$

for every real bounded martingale Y and $H\underset{c}{\cdot}M^u = H\underset{c}{\cdot}M^v$. The theorem is then proved.

5 REMARKS.

(a) If H is weakly predictable, $H \cdot X$ is well-defined when X is a projective system of semimartingales by Ustanel [15, III.4]. Moreover, $H \cdot X$ is a special semimartingale if X is a projective system of special semimartingales.

(b) If X is a projective system of square integrable martingales, $H\underset{c}{\cdot}X$ is also a square integrable martingale. If $U \varepsilon U_h(\Phi_b^1)$ absorbs the range of H, then $H\underset{c}{\cdot}X = H\underset{c}{\cdot}X^u$ may also be expressed by Definition V.1.5.

(c) $H\underset{c}{\cdot}X$ may be characterized in the following manner: $H\underset{c}{\cdot}X$ is the unique semimartingale having decomposition $H\underset{c}{\cdot}X = \langle H_0, X_0^u\rangle + W + H \cdot A^u$, such that for every real bounded martingale Y, $[W, Y] - H \cdot B^Y$ is a local martingale which is zero at 0, where $B^Y = [Y, \langle \cdot, X^u\rangle]$.

(d) We may immediately extend $H\underset{c}{\cdot}X$ for a mapping H that has a locally bounded range, provided H_0 remains bounded. For we may choose stopping times (T_n) which increase to infinity such that $H1_{[0,T_n]}$ has a bounded range. We let $U_n \varepsilon U_h(\Phi_b^1)$ absorb the range of $H1_{[0,T_n]}$. Since $H1_{[0,T_n]}$ takes values in both $\Phi[U_n^0]$ and $\Phi[U_{n+1}^0]$, we have

$$
\begin{aligned}
(H1_{[0,T_{n+1}]} \overset{\bullet}{c} X)^{T_n} &= (H1_{[0,T_{n+1}]} \overset{\bullet}{c} X^{u_{n+1}})^{T_n} \\
&= H1_{[0,T_n]} \overset{\bullet}{c} X^{u_{n+1}} \\
&= H1_{[0,T_n]} \overset{\bullet}{c} X^{u_n} \\
&= H1_{[0,T_n]} \overset{\bullet}{c} X.
\end{aligned}
$$

We then define $H \underset{c}{\bullet} X$ by pasting.

6 THEOREM. Let H be a weakly optional, Φ-valued mapping on $R^+ \times \Omega$ having a bounded range and let X be a projective system of special semimartingales.

(a) If K is a weakly optional, Φ-valued mapping on $R^+ \times \Omega$ having a bounded range, then $(H + K)\underset{c}{\bullet}X = H\underset{c}{\bullet}X + K\underset{c}{\bullet}X$.

(b) If Y is a projective system of special semimartingales, then $H\underset{c}{\bullet}(X + Y) = H\underset{c}{\bullet}X + H\underset{c}{\bullet}Y$.

(c) If T is a stopping time, then $(H\underset{c}{\bullet}X)^T = H\underset{c}{\bullet}X^T = H1_{[0,T]} \overset{\bullet}{c} X$.

Proof:

(a) Let $U \in U_h(\Phi_b^1)$ absorb the range of H, K, and H + K. Then, by Theorem V.3.1,

$$(H + K)\underset{c}{\bullet}X = (H + K)\underset{c}{\bullet}X^u$$

$$= H \underset{c}{\cdot} X^u + K \underset{c}{\cdot} X^u$$

$$= H \underset{c}{\cdot} X + K \underset{c}{\cdot} X.$$

(b) If $U \in U_h(\Phi_b^1)$ absorbs the range of H, then, by Definition 2.9 and Theorem V.3.1,

$$H \underset{c}{\cdot} (X + Y) = H \underset{c}{\cdot} (X^u + Y^u)$$

$$= H \underset{c}{\cdot} X^u + H \underset{c}{\cdot} Y^u$$

$$= H \underset{c}{\cdot} X + H \underset{c}{\cdot} Y.$$

(c) By Theorem V.3.2,

$$(H \underset{c}{\cdot} X)^T = (H \underset{c}{\cdot} X^u)^T$$

$$= H1_{[0,T]} \underset{c}{\cdot} X^u$$

$$= H1_{[0,T]} \underset{c}{\cdot} X,$$

and

$$(H \underset{c}{\cdot} X)^T = (H \underset{c}{\cdot} X^u)^T$$

$$= H \underset{c}{\cdot} (X^u)^T$$

$$= H \underset{c}{\cdot} X^T,$$

where $U \in U_h(\Phi_b^1)$ absorbs the range of H.

Using Theorem V.3.9, we immediately obtain an associativity property.

7 THEOREM. Let $X = \{X^u : U \in U_h(\Phi_b^1)\}$ be a projective system of special semimartingales.

(a) If H is a weakly optional, Φ-valued mapping on $R^+ \times \Omega$ having a bounded range and K is a real, bounded, predictable process, then $KH\underset{c}{\cdot}X = H\underset{c}{\cdot}(K\cdot X) = K\cdot(H\underset{c}{\cdot}X)$.

(b) If K is a weakly predictable, Φ-valued mapping on $R^+ \times \Omega$ having a bounded range and H is a real, bounded, optional process, then $HK\underset{c}{\cdot}X = H\underset{c}{\cdot}(K\cdot X) = K\cdot(H\underset{c}{\cdot}X)$.

Proof:

(a) Let $U \in U_h(\Phi_b^1)$ absorb the range of both H and KH. Then

$$\begin{aligned} KH\underset{c}{\cdot}X &= KH\underset{c}{\cdot}X^u \\ &= H\underset{c}{\cdot}(K\cdot X^u) \\ &= H\underset{c}{\cdot}(K\cdot X)^u \\ &= H\underset{c}{\cdot}(K\cdot X), \end{aligned}$$

and

$$\begin{aligned} KH\underset{c}{\cdot}X &= KH\underset{c}{\cdot}X^u \\ &= K\cdot(H\underset{c}{\cdot}X^u) \\ &= K\cdot(H\underset{c}{\cdot}X). \end{aligned}$$

The proof of (b) is similar.

We conclude with a convergence theorem.

8 THEOREM.

(a) Let (H^n) and H be weakly optional, Φ-valued mappings on $R^+ \times \Omega$ such that (H^n) converges pointwise to H in Φ. Suppose there exists $U \in U_h(\Phi_b^1)$ such that (H^n) and H take values in $\Phi[U^0]$. If X is a projective system of special semimartingales (resp. square integrable martingales), then $(H^n \underset{c}{\cdot} X)$ converges locally in S^1 (resp. converges in M^2) to $H \underset{c}{\cdot} X$.

(b) Let H be a weakly optional, Φ-valued mapping on $R^+ \times \Omega$ having a bounded range. Let (M^n) be a sequence of square integrable martingales in Φ^1 converging to M in $M(\Phi^1)$. Then $(H \underset{c}{\cdot} M^n)$ converges to $H \underset{c}{\cdot} M$ in M^2.

Proof:

(a) Since H^n converges to H in Φ, $\lim_n p_u(H^n - H) = 0$; hence, H^n converges to H in $\Phi[U^0]$. Then since

$$H^n \underset{c}{\cdot} X - H \underset{c}{\cdot} X = H^n \underset{c}{\cdot} X^u - H \underset{c}{\cdot} X^u,$$

the result follows from Theorem V.3.5 (resp. V.1.12).

(b) We choose $U \in U_h(\Phi_b^1)$ absorbing the range of H. Then since $(M^n)^u$ converges to M^u in $M^2_{\Phi^1(u)}$ and

$$H \underset{c}{\cdot} M^n - H \underset{c}{\cdot} M = H \underset{c}{\cdot} (M^n)^u - H \underset{c}{\cdot} M^u,$$

the result follows by Theorem V.1.13.

CHAPTER VII
SUMMARY AND CONCLUSIONS

The theory of the predictable stochastic integral has been fully developed in many settings, with many varying approaches, and the predictable σ-algebra plays an important role in both theory and applications. However, the characterization of the stochastic integral as an operator in terms of brackets, due first to Kunita and Watanabe, avoids the nature of the predictable σ-algebra and allows the extension of the stochastic integral to optional, and even measurable, processes. Meyer [4, 11] has fully developed this approach and has defined the classical compensated optional integral of a real optional process with respect to a real local martingale. Due to the geometry of Hilbert space, one may define the square bracket $[X, Y]$ of Hilbert-valued semimartingales X and Y and enjoy all the properties of the square bracket of real semimartingales. These properties allow one, in certain cases, to characterize stochastic integrals of optional processes in Hilbert space in a manner analogous to the real-valued setting. In this study, we have developed the theories of three optional stochastic integrals in Hilbert space and have applied these integrals to the theory of stochastic integration in nuclear spaces.

Using the Kunita-Watanabe inequality, we have first defined the stochastic integral of a real optional process with respect to a

Hilbert square integrable martingale. As in the real-valued case, this integral is defined in terms of a continuous linear operator on L^2. We have then extended this integral to Hilbert special semimartingales, using Meyer's method for real processes, and have characterized the optional stochastic integral in terms of the square bracket of Hilbert semimartingales.

Secondly, we have defined the stochastic integral of an optional Hilbert-valued process with respect to a real square integrable martingale. We constructed this integral by means of an orthonormal expansion of the optional process and then showed, as a consequence, that the integral defines a Hilbert-valued, compact, linear operator on L^2. We then made the extension to integrating all locally bounded, optional, Hilbert-valued processes with respect to real special semimartingales.

Lastly, we defined the stochastic integral of a locally bounded, optional, Hilbert-valued process with respect to a Hilbert-valued special semimartingale. This integral is again characterized in terms of the square bracket. The approach in this setting begins by constructing the integral of a bounded optional process with respect to a square integrable martingale. In this construction, we obtained a method that also applies to the first two optional integrals and in particular to the classical optional integral of real processes. More importantly, we obtained a natural definition of the optional stochastic integral in terms of the predictable stochastic integral.

In each setting, the integrals defined are true compensated stochastic integrals, which are extensions of the predictable stochastic integral, and which agree with the classical optional integral when the Hilbert space is the real line. The integrals have been shown to have all of the properties of the classical optional integral and to satisfy convergence theorems.

The theory of stochastic integration in Hilbert space is the first and most natural extension of the stochastic integral of real-valued processes. Kunita [8] was the first to consider the predictable stochastic integral in Hilbert space. We have followed his example by defining the optional stochastic integral in Hilbert space for several different settings. Using these integrals, we have further developed Ustanel's [15] theory of stochastic integration in nuclear spaces by defining three optional integrals in nuclear space. The theory of the predictable stochastic integral in Hilbert spaces has since been widely developed and the general theory in Banach spaces continues to expand. We hope that this study serves as a foundation of the continued development of the optional stochastic integral in abstract spaces.

APPENDIX
EQUIVALENCE OF THE BMO SPACES

We provide here the details for showing the equivalence of the BMO^p spaces of E-valued martingales, for $p \geq 1$. The consequence of this result is that H^1 is complete, which we shall also prove. Lastly, we shall define the S^p spaces of semimartingales.

1 LEMMA. Let Y be in BMO^p.

(a) For all stopping times S and T such that $S \leq T$,

$E(|Y_T - Y_{S-}|^p | F_S) \leq (\|Y\|_{BMO^p})^p$ a.s., and

(b) $|\Delta Y| \leq \|Y\|_{BMO^p}$.

Proof:

(a) Let $c = \|Y\|_{BMO^p}$. Then almost surely

$$\begin{aligned} E(|Y_T - Y_{S-}|^p|F_S) &= E(|E(Y_\infty|F_T) - E(Y_{S-}|F_T)|^p|F_S) \\ &= E(|E(Y_\infty - Y_{S-}|F_T)|^p|F_S) \\ &\leq E(E(|Y_\infty - Y_{S-}|^p|F_T)|F_S) \\ &= E(|Y_\infty - Y_{S-}|^p|F_S) \\ &\leq c^p. \end{aligned}$$

(b) We let T = S in part (a). Then

$$|\Delta Y_S|^p \leq (\|Y\|_{BMO^p})^p \text{ a.s.,}$$

for every stopping time S. Hence, by the cross-section theorem, $|\Delta Y| \leq \|Y\|_{BMO^p}$.

2 LEMMA. If $Y \in BMO^p$, then for every stopping time S, $E(|Y_\infty - Y_S|^p|F_S) \leq (\|Y\|_{BMO^p})^p$ a.s.

Proof: We let $S_n = S + \frac{1}{n}$. By right continuity, Y_{S_n-} converges to Y_S. Hence, by Fatou's lemma, we have almost surely

$$\begin{aligned} E(|Y_\infty - Y_S|^p|F_S) &\leq \liminf_n E(|Y_\infty - Y_{S_n-}|^p|F_S) \\ &= \liminf_n E(E(|Y_\infty - Y_{S_n-}|^p|F_{S_n})|F_S) \\ &\leq \liminf_n E((\|Y\|_{BMO^p})^p|F_S) \\ &= (\|Y\|_{BMO^p})^p. \end{aligned}$$

3 THEOREM. Let Y be in BMO^1. For all nonnegative constants a and b, we have

$$bP\{Y^* \geq a + b\} \leq \|Y\|_{BMO^1} P\{Y^* \geq a\}.$$

Proof: Clearly, we may assume $b > 0$. Let $c = \|Y\|_{BMO^1}$. We define stopping times S and T by $S = \inf\{t: |Y_t| \geq a\}$ and $T = \inf\{t: |Y_t| \geq a + b\}$. Then $|Y_{S-}| \leq a$ on all of Ω.

Let $A = \{Y^* > a + b\}$. For every w in A, there exists t such that $|Y_t(w)| > a + b$; hence, the sets $\{t: |Y_t(w)| \geq a\}$ and $\{t: |Y_t(w)| \geq a + b\}$ are nonempty. Thus, by right continuity, $|Y_S| \geq a$ and $|Y_T| \geq a + b$ on the set A. Moreover, if we let $B = \{|Y_S| \geq a\}$, then $A \subseteq B$ and $B \subseteq \{Y^* \geq a\}$.

Since $|Y_{S-}| - a \leq 0$, we have on A

$$\begin{aligned} a + b &\leq |Y_T| \\ &\leq |Y_T - Y_{S-}| + |Y_{S-}|; \end{aligned}$$

hence,

$$b \leq |Y_T - Y_{S-}|.$$

Since, by Lemma 1, $E(|Y_T - Y_{S-}| \,|\, F_S) \leq c$ a.s. and $B \in F_S$, we then have

$$\begin{aligned} bP(Y^* > a + b) &= \int_A b\,dP \\ &\leq \int_A |Y_T - Y_{S-}|\,dP \\ &\leq \int_B |Y_T - Y_{S-}|\,dP \\ &\leq \int_B c\,dP \\ &\leq cP(Y^* \geq a). \end{aligned}$$

We now replace b by a sequence (b_n), which increases to b, with $b_n > 0$ for each n. Then

$$b_nP\{Y^* > a + b_n\} \leq cP\{Y^* \geq a\}.$$

Letting n tend to infinity, we obtain

$$bP\{Y^* \geq a + b\} \leq cP\{Y^* \geq a\}.$$

4 COROLLARY. Let Y be in BMO^1. Then for every $b \geq 0$, $bE[Y^* - b] \leq \|Y\|_{BMO^1}E[Y^*]$.

Proof: We again let $c = \|Y\|_{BMO^1}$. By Theorem 3, we have for every $t \geq 0$

$$\begin{aligned} b\int 1_{\{Y^* \geq t+b\}}dP &= bP\{Y^* \geq t + b\} \\ &\leq cP\{Y^* \geq t\} \\ &= c\int 1_{\{Y^* \geq t\}}dP. \end{aligned}$$

Integrating with respect to t and applying Fubini's theorem, we obtain

$$\begin{aligned} bE[Y^* - b] &\leq bE[(Y^* - b)^+] \\ &= b\int\int 1_{\{Y^*-b \geq t\}}dtdP \\ &= b\int\int 1_{\{Y^* \geq t+b\}}dPdt \\ &\leq c\int\int 1_{\{Y^* \geq t\}}dPdt \end{aligned}$$

$$= c\int\int 1_{\{Y^*\geq t\}}\,dt\,dP$$

$$= cE[Y^*].$$

5 THEOREM. Let Y be in BMO^1. Then $E[Y^*] \leq 4\|Y\|_{BMO^1}$.

Proof: If we assume $E[Y^*] < \infty$, we may apply Corollary 4. Letting $c = \|Y\|_{BMO^1}$ and $b = 2c$, we have

$$2cE[Y^* - 2c] \leq cE[Y^*].$$

Hence, $E[Y^*] \leq 4c$.

Otherwise, we define stopping times T_n, which increase to infinity, by $T_n = \inf\{t: |Y_t| \geq n\}$. Then Y^{T_n} is also a uniformly integrable martingale. We shall show that Y^{T_n} is in BMO^1. Since Y is in BMO^1, Y has jumps bounded by $c = \|Y\|_{BMO^1}$, by Lemma 1. Thus, for every stopping time S, we have a.s.

$$\begin{aligned} E(|Y_\infty^{T_n} - Y_{S-}^{T_n}||F_S) &\leq E(|Y_\infty^{T_n} - Y_S^{T_n}||F_S) + E(|\Delta Y_S^{T_n}||F_S) \\ &\leq E(|E(Y_\infty - Y_{T_n\wedge S}|F_{T_n})||F_S) + c \\ &\leq E(E(|Y_\infty - Y_{T_n\wedge S}||F_{T_n})|F_S) + c \\ &= E(|Y_\infty - Y_{T_n\wedge S}||F_{T_n\wedge S}) + c \\ &\leq 2c. \end{aligned}$$

The last inequality is Lemma 2.

Thus, Y^{T_n} is in BMO^1 and $\|Y^{T_n}\|_{BMO^1} \leq 2c$, for each n. Moreover, since Y_{T_n} is integrable and

$$(Y^{T_n})^* \leq \sup_{s<T_n} |Y_s| \vee |Y_{T_n}|$$

$$\leq n \vee |Y_{T_n}|,$$

we have $E[(Y^{T_n})^*] < \infty$.

Hence, by the first line of the proof, we have

$$E[(Y^{T_n})^*] \leq 4(2c) = 8c < \infty.$$

By the monotone convergence theorem, we have $E[Y^*] \leq 8c < \infty$. We may then apply the first line of the proof and we have, in fact, $E[Y^*] \leq 4c$.

6 THEOREM. Let Y be in BMO^1. Then for every $n \geq 1$ $E[(Y^*_\infty)^n] \leq n!(8\|Y\|_{BMO^1})^n$.

Proof: We again let $c = \|Y\|_{BMO^1}$. Let A be in F_0 such that $P(A) > 0$. We then replace P by the law $Q = \frac{1}{P(A)}(1_A P)$ and apply Theorem 5, with the law Q, to obtain $E(Y^*|F_0) \leq 4c$ a.s.

Let S be a stopping time. We consider the process $X_t = Y_{S+t} - Y_{S-}$ with respect to the filtration $(G_t) = (F_{S+t})$. Then

$X_\infty = Y_\infty - Y_{S-}$ is integrable and for every stopping time T, with respect to (G_t),

$$E[X_T] = E[Y_{S+T}] - E[Y_{S-}]$$

is constant, since S + T is a stopping time with respect to (F_t). Hence, X is a uniformly integrable martingale. Moreover, we have almost surely

$$\begin{aligned} E(|X_\infty - X_{T-}||G_T) &\le E(|X_\infty - X_T|G_T) + E(|\Delta X_T||G_T) \\ &= E(|Y_\infty - Y_{S+T}||F_{S+T}) + E(|\Delta Y_{S+T}||F_{S+T}) \\ &\le 2c, \end{aligned}$$

by Lemmas 1 and 2.

Hence, X is in BMO^1 and $\|X\|_{BMO^1} \le 2c$. Thus, by Theorem 5, $E[X^*] \le 8c$. By conditioning on G_0, as in the beginning of the proof, we obtain $E(X^*|G_0) \le 8c$ a.s. Hence,

$$E(\sup_{S\le t}|Y_t - Y_{S-}||F_S) \le 8c.$$

We then have, for every stopping time S,

$$\begin{aligned} Y^*_\infty - Y^*_{S-} &= \sup_t|Y_t| - \sup_{t<S}|Y_t| \\ &\le \sup_{S\le t}|Y_t - Y_{S-}|; \end{aligned}$$

hence,

$$E(Y^*_\infty - Y^*_{S-}|F_S) \leq 8c \text{ a.s.}$$

Since Y^* is an increasing process, we obtain [4, VI.105 (a)]

$$E[(Y^*_\infty)^n] \leq n!(8c)^n, \text{ for } n \geq 1.$$

7 THEOREM. All BMO^p spaces are equivalent, for $p \geq 1$.

Proof: Let $1 \leq p \leq q$. Since $p/q < 1$, we have for every stopping time S and for every Y in BMO^q

$$\begin{aligned} E(|Y_\infty - Y_{S-}|^p|F_S) &= E((|Y_\infty - Y_{S-}|^q)^{p/q}|F_S) \\ &\leq (E(|Y_\infty - Y_{S-}|^q|F_S)^{p/q} \\ &\leq (\|Y\|^q_{BMO^q})^{p/q} \text{ a.s.} \\ &= \|Y\|^p_{BMO^q}. \end{aligned}$$

Hence, Y is in BMO^p and $\|Y\|_{BMO^p} \leq \|Y\|_{BMO^q}$.

To complete the proof, it suffices to show, for each $n \geq 1$, $BMO^1 \subseteq BMO^n$ and $\|Y\|_{BMO^n} \leq k\|Y\|_{BMO^1}$ for some constant k.

Let Y be in BMO^1 and let $c = \|Y\|_{BMO^1}$. By Theorem 6, we have $E[(Y^*_\infty)^n] \leq n!(8c)^n$, for $n \geq 1$. By conditioning on F_0, we obtain $E((Y^*_\infty)^n|F_0) \leq n!(8c)^n$ a.s.

Let S be a stopping time and let $X_t = Y_{S+t} - Y_{S-}$, with respect to the filtration $(G_t) = (F_{S+t})$. Then, as in Theorem 6, X is in BMO^1 and

$d = \|X\|_{BMO^1} \leq 2c$. Hence, $E[(X_\infty^*)^n] \leq n!(8d)^n$ and, by conditioning on G_0, we have for $n \geq 1$,

$$E((X_\infty^*)^n | G_0) \leq n!(8d)^n \text{ a.s.}$$

That is, we have a.s.

$$E((\sup_{S\leq t}|Y_t - Y_{S-}|)^n | F_S) \leq n!(16c)^n.$$

Finally, we obtain, for $n \geq 1$,

$$\begin{aligned} E(|Y_\infty - Y_{S-}|^n | F_S) &\leq E((\sup_{S\leq t}|Y_t - Y_{S-}|)^n | F_S) \\ &\leq n!(16c)^n \text{ a.s.} \\ &= ((n!)^{1/n} 16c)^n. \end{aligned}$$

Hence, Y is in BMO^n and $\|Y\|_{BMO^n} \leq (n!)^{1/n} 16 \|Y\|_{BMO^1}$, and the theorem is proved.

We note, in particular, that

$$(7.1) \qquad \|Y\|_{BMO^1} \leq \|Y\|_{BMO^2} \leq 16\sqrt{2}\, \|Y\|_{BMO^1}.$$

We now recall that H^1 is the space of E-valued local martingales X such that $E[[X, X]_\infty^{1/2}] < \infty$. Brooks and Dinculeanu [1] have defined H_E^1 to be the space of E-valued (in general, Banach valued) martingales

X such that $E[X^*] < \infty$. They show H^1_E is complete and its dual is BMO^1. We have seen, in Theorem II.5.8, that the dual of H^1 is BMO^2. To show that H^1 is complete, we shall show that H^1 is equivalent to H^1_E.

8 THEOREM. The space H^1 is equivalent to H^1_E.

Proof: Let $q(X) = E[[X, X]_\infty^{1/2}]$ and $m(X) = E[X^*]$. Let X be a bounded martingale and let $\alpha = \sup\limits_{\substack{Y \in BMO \\ \|Y\|_{BMO^1} \leq 1}} |E[(X_\infty, Y_\infty)]|$ We assert that $\frac{1}{32}\alpha \leq q(x) \leq 3\alpha$ and $\frac{1}{10}\alpha \leq m(X) \leq 2\alpha$.

Let J be a continuous linear form on H^1 with $\|J\| \leq 1$. By Theorem II.5.8, there exists Y in BMO^2 such that $\|Y\|_{BMO^2} \leq 3$ and $J(W) = E[(W_\infty, Y_\infty)]$ for every W in H^1. Let $Z = \frac{1}{3}Y$. Then, Z is still in BMO^2, and hence in BMO^1, and

$$\|Z\|_{BMO^1} \leq \|Z\|_{BMO^2} \leq 1.$$

Thus,

$$|J(X)| = |E[(X_\infty, Y_\infty)]| = 3|E[(X_\infty, Z_\infty)]| \leq 3\alpha.$$

Hence, $q(X) \leq 3\alpha$.

Let Y be in BMO^1 with $\|Y\|_{BMO^1} \leq 1$. Then $\|Y\|_{BMO^2} \leq 16\sqrt{2}$. Thus, the continuous, linear form J defined on H^1 by

$$J(W) = E[(W_\infty, Y_\infty)],$$

for every W in H^1, has norm ≤ 32, since by Corollary II.5.7, $|J(W)| \leq \sqrt{2}\,\|Y\|_{BMO^2}\|W\|_{H^1}$. Letting $L = \frac{1}{32} J$, we have $\|L\| \leq 1$; hence,

$$\begin{aligned} |E[(X_\infty, Y_\infty)]| &= |J(X)| \\ &= 32|L(X)| \\ &\leq 32q(x). \end{aligned}$$

Thus, $\alpha \leq 32q(x)$.

If J is a continuous linear form on H^1_E with $\|J\| \leq 1$, there exists Y in BMO^1 such that $J(W) = E[(W_\infty, Y_\infty)]$, for every bounded W in H^1_E, and $\|Y\|_{BMO^1} \leq 2$ [1, 3.7]. As above, we have $|J(X)| \leq 2\alpha$ and hence $m(X) \leq 2\alpha$.

If Y is in BMO^1 and $\|Y\|_{BMO^1} \leq 1$, then there exists a continuous linear form J on H^1_E such that $J(W) = E[(W_\infty, Y_\infty)]$ for bounded W [1, 3.7]. Moreover, $\|J\| \leq 10$. (Brooks and Dinculeanu [1] showed that $|||J||| \leq 10$; however, since J is a real valued operator, we have $\|J\| = |||J|||$.) As above, it follows that $|E[(X_\infty, Y_\infty)]| \leq 10m(X)$ and hence that $\alpha \leq 10m(X)$.

Thus, for bounded X, we have $\frac{1}{32}\alpha \leq q(X) \leq 3\alpha$ and $\frac{1}{10}\alpha \leq m(X) \leq 2\alpha$. Hence,

$$q(X) \leq 30m(X) \tag{8.1}$$

and

(8.2) $$m(X) \leqq 64q(X)$$

for bounded X.

Let X be a local martingale. We define predictable stopping times (T_n) which increase to infinity by

$$T_n = \inf\{t: X^*_{t-} \geq n\}.$$

Then $Y^n = X^{T_n-}$ is a bounded local martingale; thus, it is a true bounded martingale. Since T_n increases to infinity, we have, by monotone convergence and Theorem II.3.8,

$$\begin{aligned} q(X) &= E[[X, X]^{1/2}_\infty] \\ &= \lim_n E[[X, X]^{1/2}_{T_n-}] \\ &= \lim_n E[[Y^n, Y^n]^{1/2}_\infty] \\ &= \lim_n q(Y^n). \end{aligned}$$

Again by monotone convergence, we have

$$\begin{aligned} m(X) &= E[X^*] \\ &= \lim_n E[(Y^n)^*] \\ &= \lim_n m(Y^n). \end{aligned}$$

Since Y^n are bounded, we obtain (8.1) and (8.2) for any local martingale X. It follows that H^1 and H^1_E are equivalent.

We conclude with some comments on S^p spaces of semimartingales.

9 DEFINITION. Let $X = X_0 + M + A$ be a semimartingale, where M is a local martingale, A is a process of finite variation, and $M_0 = 0 = A_0$. We let, for $p \geq 1$,

$$j_p(M, A) = \|[M, M]_\infty^{1/2}\|_{L^p} + \|\int^\infty |dA_s|\|_{L^p}.$$

We say X is in S^p if

$$\|X\|_{S^p} = \|X_0\|_{L^p} + \inf_{M,A} j_p(M, A) < \infty,$$

where the infimum is taken over all possible decompositions of X.

10 REMARKS.

(a) We note that there are several other equivalent definitions of S^p. In particular, we may attribute the value of X_0 to M and say $\|X\|_{S^p} = \inf_{M,A} j_p(M, A)$.

(b) Suppose X is a special semimartingale with canonical decomposition $X = X_0 + M + A$. Let H be a locally bounded, optional process. (We may take H and X to be either real or E-valued.) Then

$$H \underset{c}{\cdot} X = H_0 X_0 + H \underset{c}{\cdot} M + H \cdot A$$

is one decomposition of the semimartingale $H_{\dot{c}}X$. Hence,

$$\|H_{\dot{c}}X\|_{S^1} \leq \|H_0X_0\|_{L^1} + \|[H_{\dot{c}}M,\ H_{\dot{c}}M]_\infty^{1/2}\|_{L^1} + E[\int^\infty |d(H\cdot A)_s|]$$

$$= \|H_0X_0\|_{L^1} + \|H_{\dot{c}}M\|_{H^1} + E[\int^\infty |H_s||dA_s|].$$

(c) Let X be a special semimartingale with canonical decomposition $X = X_0 + M + A$. Define $\|X\|$ by

$$\|X\| = \|X_0\|_{L^1} + \|M\|_{H^1} + E[\int^\infty |dA_s|].$$

Then clearly $\|X\|_{S^1} \leq \|X\|$. On the other hand, we assert $\|X\| \leq 3\|X\|_{S^1}$.

Let $X = X_0 + N + B$ be any other decomposition of X. Since X is special, B is locally integrable. Hence,

$$X = X_0 + (N + B - B^p) + B^p.$$

Since the canonical decomposition is unique, we have $M = N + B - B^p$ and $A = B^p$.

By the proof of Corollary IV.4.6, we have

$$E[\int^\infty |dB_s^p|] = E[\int^\infty |dB_s|].$$

By Theorem II.5.2, we have

$$\|B - B^p\|_{H^1} \leq E[\int^\infty |d(B_s - B_s^p)|]$$

$$\leq 2E[\int^\infty |dB_s|].$$

Hence,

$$\begin{aligned} \|X\| &= \|X_0\|_{L^1} + \|N + B - B^p\|_{H^1} + E[\int^\infty |dB_s^p|] \\ &\leq \|X_0\|_{L^1} + \|N\|_{H^1} + \|B - B^p\|_{H^1} + E[\int^\infty |dB_s|] \\ &\leq \|X_0\|_{L^1} + \|N\|_{H^1} + 3E[\int^\infty |dB_s|] \\ &\leq 3(\|X_0\|_{L^1} + j_1(N, B)). \end{aligned}$$

Taking the infimum over such decompositions $X = X_0 + N + B$, we obtain

$$\|X\| \leq 3\|X\|_{S^1}.$$

It follows that if (X^n) is a sequence of special semimartingales, with canonical decompositions $X^n = X_0^n + M^n + A^n$, converging to a special semimartingale $X = X_0 + M + A$, then X_0^n converges to X_0 in L^1, M^n converges to M in H^1, and $\lim_{n\to\infty} E[\int^\infty |d(A_s^n - A_s)|] = 0$.

REFERENCES

[1] J.K. Brooks and N. Dinculeanu, H^1 and BMO Spaces of Abstract Martingales. Seminar on Stochastic Processes, 1985, Birkhäuser, Boston, 1986.

[2] J.K. Brooks and N. Dinculeanu, Projections and Regularity of Abstract Processes. Stochastic Analysis and Applications, 5(1) (1987), 17-25.

[3] C. Déllacherie and P. Meyer, Probabilities and Potential. Hermann, Paris, 1978.

[4] C. Déllacherie and P. Meyer, Probabilities and Potential B. North Holland, Amsterdam-New York-Oxford, 1982.

[5] J. Diestel and J.J. Uhl, Jr., Vector Measures, Mathematical Surveys, Number 15. American Mathematical Society, Providence, R.I., 1977.

[6] N. Dinculeanu, Vector Measures. Pergamon Press, New York, 1967.

[7] N. Dunford and J.T. Schwartz, Linear Operators Part I. Interscience, New York, 1957.

[8] H. Kunita, Stochastic Integrals Based on Martingales Taking Values in Hilbert Space. Nagoya Mathematics Journal, 38 (1970), 41-52.

[9] M. Métivier, Semimartingales. Walter de Gruyter, Berlin, 1982.

[10] M. Métivier and J. Pellaumail, Stochastic Integration. Academic Press, New York, 1980.

[11] P. Meyer, Un Cours sur les Intégrales Stochastiques. Séminaire de Probabilités X, Lecture Notes in Mathematics, 511. Springer-Verlag, Berlin-Heidelberg-New York, 1976.

[12] A. Pietsch, Nuclear Locally Convex Spaces. Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[13] L. Schwartz, Semimartingales and their Stochastic Calculus on Manifolds. Les Presses de l'Université de Montréal, Montreal, 1984.

[14] F. Treves, Topological Vector Spaces, Distributions, and Kernels. Academic Press, New York-London, 1967.

[15] S. Ustanel, Stochastic Integration on Nuclear Spaces and Its Applications. Annales de l'Institut Henri Poincaré, 18(2) (1982), 165-200.

[16] M. Yor, En Cherchant une Définition Naturelle des Intégrales Stochastiques Optionnelles. Séminaire de Probabilités XIII, Lecture Notes in Mathematics, 721. Springer-Verlag, Berlin-Heidelberg-New York, 1979.

BIOGRAPHICAL SKETCH

David Neal was born and raised in San Antonio, Texas. He became interested in mathematics at a young age and participated in math contests throughout middle school and high school. He moved to Florida in 1979 and enrolled at the University of Florida in 1980, earning B.S. and M.S. degrees in mathematics in 1983 and 1985, respectively. David is a member of Phi Beta Kappa and his hobbies include music, reading, sports, chess, and astronomy.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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